Electricity Markets: Summer Semester 2016, Lecture 2

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Short-run Efficient Operation of Electricity Markets

Efficient Markets for the short-run

Assume investments already made in generators and and consumption assets (factories, machines, etc.).

Assume all actors are price takers (i.e. nobody can exercise market power) and we have perfect competition.

How do we allocate production and consumption in the most efficient way?

I.e. we are interested in the short-run "static" efficiency.

Electricity Markets from the Consumer Perspective

Consumer behaviour: Theory

Suppose for some given period a consumer consumes electricity at a rate of Q MW.

Their utility or value function $U(Q)$ in €/h is a measure of their benefit for a given consumption rate Q.

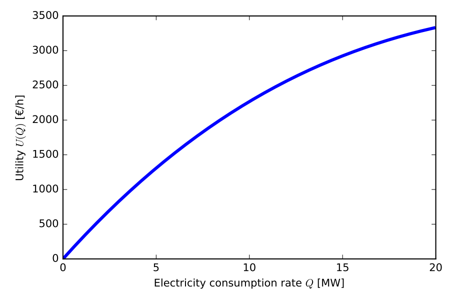
For a firm this could be the profit related to this electricity consumption from manufacturing goods.

Typical the consumer has a higher utility for higher Q, i.e. the first derivative is positive $U'(Q) > 0$. By assumption, the rate of value increase with consumption decreases the higher the rate of consumption, i.e. $U''(Q) < 0$.

Utility: Example

A widget manufacturer has a utility function which depends on the rate of electricity consumption Q [€/h] as

$$U(Q) = 0.0667\ Q^3 - 8\ Q^2 + 300\ Q$$



Note that the slope is always positive, but becomes less positive for increasing Q.

Optimal consumer behaviour

We assume to begin with that the consumer is a price-taker, i.e. they cannot influence the price by changing the amount they consume.

Suppose the market price is λ €/MWh. The consumer should adjust their consumption rate Q to maximise their net surplus

$$\max_Q [U(Q) - \lambda Q]$$

This optimisation problem is optimised for $Q = Q^*$ where

$$U'(Q^*) \equiv \frac{dU}{dQ}(Q^*) = \lambda$$

[Check units: $\frac{dU}{dQ}$ has units $\frac{\text{€/h}}{\text{MW}} =$ €/MWh.]

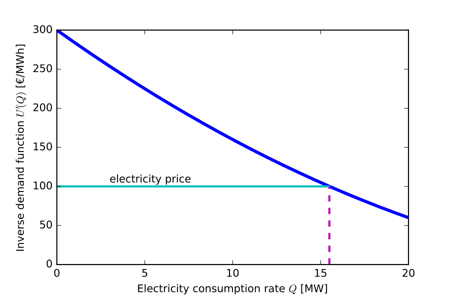
I.e. the consumer increases their consumption until they make a net loss for any increase of consumption.

$U'(Q)$ is known as the inverse demand curve or marginal utility curve, which shows, for each rate of consumption Q what price λ the consumer should be willing to pay.

Inverse demand function: Example

For our example the inverse demand function is given by

$$U'(Q) = 0.2\ Q^2 - 16\ Q + 300$$



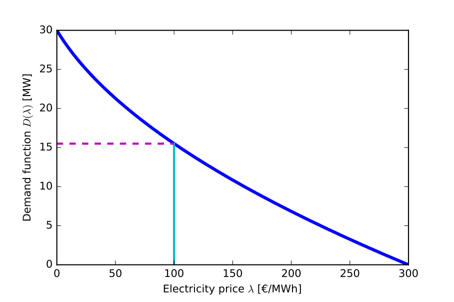
It's called the *inverse* demand function, because the demand function is the function you get from reversing the axes.

Inverse demand function: Example

The demand function $D(\lambda)$ gives the demand Q as a function of the price λ. $D(U'(Q)) = Q$.

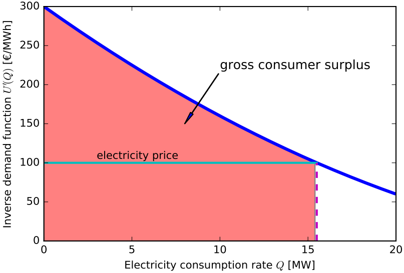
For our example the demand function is given by

$$D(\lambda) = -((\lambda + 20)/0.2)^{0.5} + 40$$



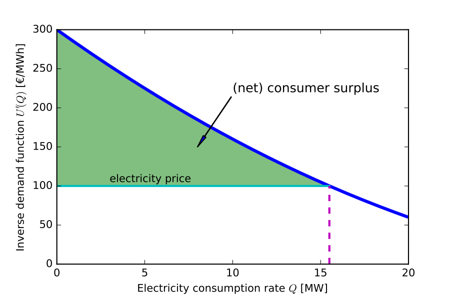
Gross consumer surplus

The area under the inverse demand curve is the gross consumer surplus, which as the integral of a derivative, just gives the utility function $U(Q)$ again, up to a constant.



Net consumer surplus

The more relevant net consumer surplus, or just consumer surplus is the net gain the consumer makes by having utility above the electricity price.

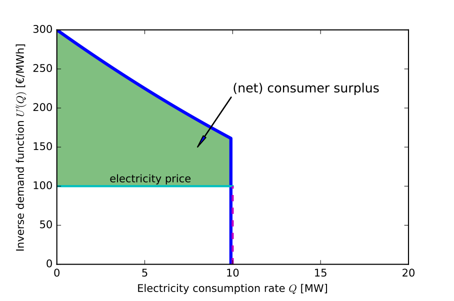


Limits to consumption

Note that it is quite common for consumption to be limited by other factors before the electricity price becomes too expensive, e.g. due to the size of electrical machinery. This gives an upper bound

$$Q \leq Q^{\max}$$

In the following case the optimal consumption is at $Q^* = Q^{\max} =$ 10 MW. We have a binding constraint and can define a shadow price μ, which indicates the benefit of relaxing the constraint $\mu^* = U'(Q^*) - \lambda$.

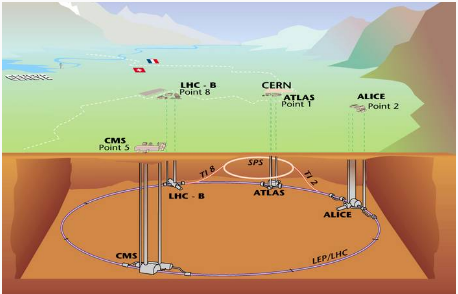


Consumers can delay their consumption

Besides changing the amount of electricity consumption, consumers can also shift their consumption in time.

For example electric storage heaters use cheap electricity at night to generate heat and then store it for daytime.

The LHC particle accelerator does not run in the winter, when prices are higher (see `http://home.cern/about/engineering/powering-cern`). Summer demand: 200 MW, corresponds to a third of Geneva, equal to peak demand of Rwanda (!); winter only 80 MW.



Source: CERN

Consumers can also move location

Aluminium smelting is an electricity-intensive process. Aluminium smelters will often move to locations with cheap and stable electricity supplies, such as countries with lots of hydroelectric power. For example, 73% of Iceland's total power consumption in 2010 came from aluminium smelting.

Aluminium costs around US$ 1500/ tonne to produce.

Electricity consumption: 15 MWh/tonne.

At Germany consumer price of €300 / MWh, this is €4500 / tonne. Uh-oh!!!

If electricity is 50% of cost, then need $750/tonne to go on electricity $\Rightarrow$ 750/15 \$/MWh = 50 \$/MWh.

Electricity Markets from the Generator Perspective

Generator Cost Function

Optimal generator behaviour follows a similar schema to that for consumers.

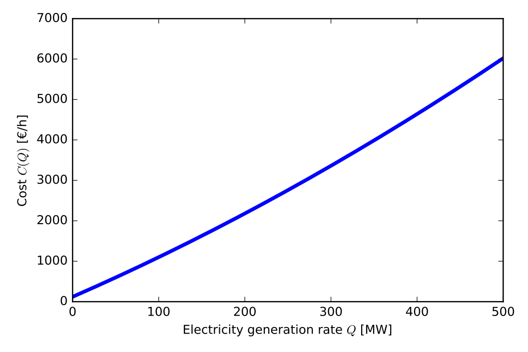
A generator has a cost or supply function $C(Q)$ in €/h, which gives the costs (of fuel, etc.) for a given rate of electricity generation Q MW.

Typical the generator has a higher cost for a higher rate of generation Q, i.e. the first derivative is positive $C'(Q) > 0$. For most generators the rate at which cost increases with rate of production itself increases as the rate of production increases, i.e. $C''(Q) > 0$.

Cost Function: Example

A gas generator has a cost function which depends on the rate of electricity generation Q [€/h] according to

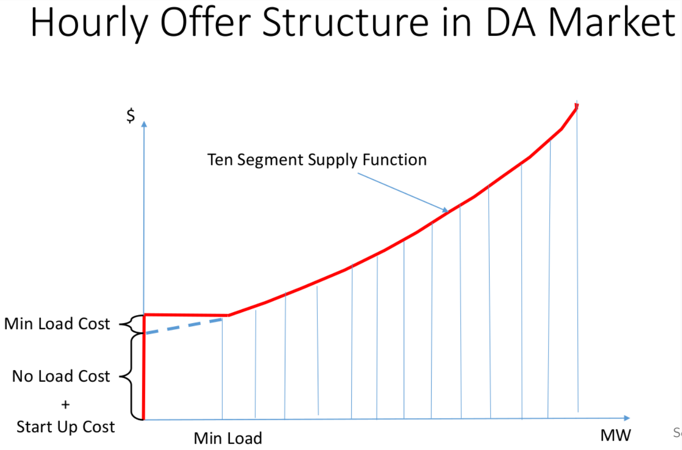
$$C(Q) = 0.005\ Q^2 + 9.3\ Q + 120$$



Note that the slope is always positive and becomes more positive for increasing Q. The curve does not start at the origin because of startup costs, no load costs, etc.

Real Example: California Day Ahead (DA) Market

Generators in California provide supply curves to the market operator as a piecewise linear function with ten segments:



Source: CAISO

Optimal generator behaviour

We assume to begin with that the generator is a price-taker, i.e. they cannot influence the price by changing the amount they generate.

Suppose the market price is λ €/MWh. For a generation rate Q, the revenue is λQ and the generator should adjust their generation rate Q to maximise their net generation surplus, i.e. their profit:

$$\max_Q [\lambda Q - C(Q)]$$

This optimisation problem is optimised for $Q = Q^*$ where

$$C'(Q^*) \equiv \frac{dC}{dQ}(Q^*) = \lambda$$

[Check units: $\frac{dC}{dQ}$ has units $\frac{\text{€/h}}{\text{MW}} = \text{€/MWh}$.]

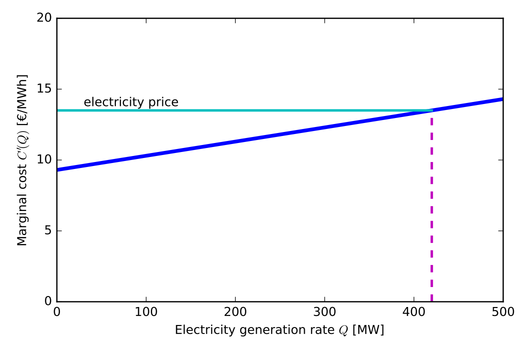
I.e. the generator increases their output until they make a net loss for any increase of generation.

$C'(Q)$ is known as the marginal cost curve, which shows, for each rate of generation Q what price λ the generator should be willing to supply at.

Marginal cost function: Example

For our example the marginal function is given by

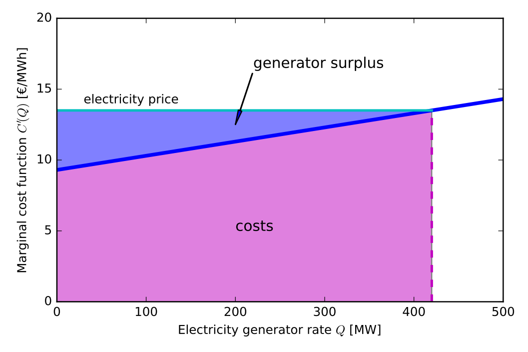
$$C'(Q) = 0.001\ Q + 9.3$$



Generator surplus

The area under the curve is generator costs, which as the integral of a derivative, just gives the cost function $C(Q)$ again, up to a constant.

The generator surplus is the profit the generator makes by having costs below the electricity price.

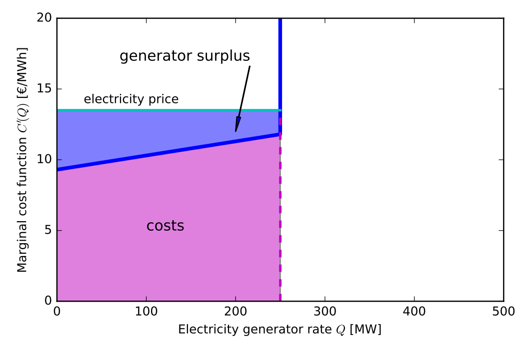


Limits to generation

Note that it is quite common for generators to be limited by e.g. their capacity, which may become a binding, i.e. limiting factor before the price plays a role, e.g.

$$Q \leq Q^{\max}$$

In the following case the optimal generation is at $Q^* = Q^{\max} =$ 250 MW. We have a binding constraint and can define a shadow price μ, which indicates the benefit of relaxing the constraint $\mu^* = \lambda - C'(Q^*)$.

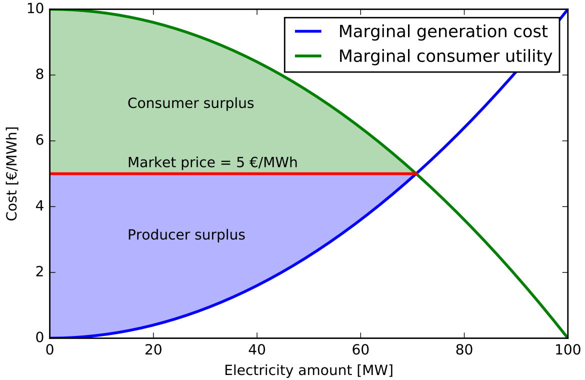


Putting generators and consumers together

Spoiler: Setting the quantity and price

Total welfare (consumer and generator surplus) is maximised if the total quantity is set where the marginal cost and marginal utility curves meet.

If the price is also set from this point, then the individual optimal actions of each actor will achieve this result in a perfect decentralised market.



Spoiler: The result of optimisation

This is the result of maximising the total economic welfare, the sum of the consumer and the producer surplus for consumers with consumption Q_i^B and generators generating with rate Q_i^S:

$$\max_{\{Q_i^B\},\{Q_i^S\}} \left[\sum_i U_i(Q_i^B) - \sum_i C_i(Q_i^S) \right]$$

subject to the supply equalling the demand in the balance constraint:

$$\sum_i Q_i^B - \sum_i Q_i^S = 0 \qquad \leftrightarrow \qquad \lambda$$

and any other constraints (e.g. limits on generator capacity, etc.).

Market price λ is the shadow price of the balance constraint, i.e. the cost of supply an extra increment 1 MW of demand.

Why decentralised markets work (in theory)

We will now show our main result:

Welfare-maximisation through decentralised markets

The welfare-maximising combination of production and consumption can be achieved by the decentralised profit-maximising decisions of producers and the utility-maximising decisions of consumers, provided that:

- The market price is equal to the constraint marginal value of the overall supply-balance constraint in the welfare maximisation problem
- All producers and consumers are price-takers

Supply-demand example: Generator bids

Example from Kirschen and Strbac pages 56-58.

The following generators bid into the market for the hour between 0900 and 1000 on 20th April 2016:

Company	Quantity [MW]	Price [$/MWh]
Red	200	12
Red	50	15
Red	150	20
Green	150	16
Green	50	17
Blue	100	13
Blue	50	18

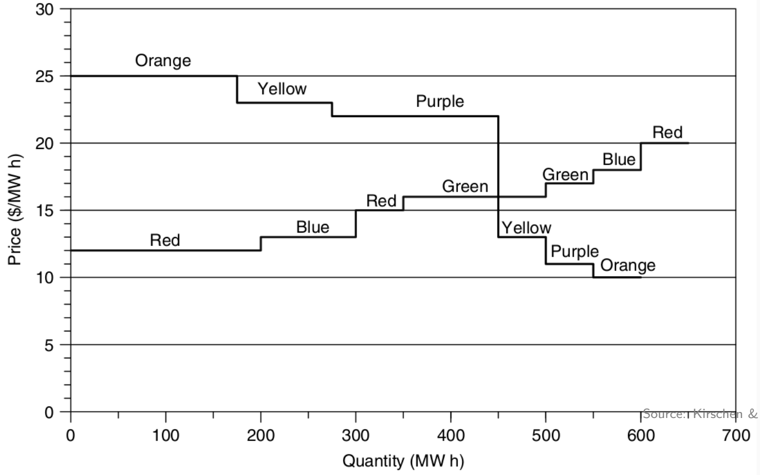
Supply-demand example: Consumer offers

The following consumers make offers for the same period:

Company	Quantity [MW]	Price [$/MWh]
Yellow	50	13
Yellow	100	23
Purple	50	11
Purple	150	22
Orange	50	10
Orange	200	25

Supply-demand example: Curve

If the bids and offers are stacked up in order, the supply and demand curves meet with a demand of 450 MW at a system marginal price of $\lambda = 16$ \$/MWh.



Source: Kirschen & Strbac

Supply-demand example: Revenue and Expenses

Dispatch and revenue/expense of each company:

Company	Production [MWh]	Consumption [MWh]	Revenue [$]	Expense [$]
Red	250		4000	
Blue	100		1600	
Green	100		1600	
Orange		200		3200
Yellow		100		1600
Purple		150		2400
Total	450	450	7200	7200

Several generators with fixed demand

Consider a simplified situation where we have just two generators 1 and 2 which have to meet a fixed demand Q (could be e.g. many consumers with inelastic demand and inverse demand curve much higher than the generation costs, i.e. $U(Q) = d \cdot Q$ where $d >> c_i$).

They have generation rate Q_1 and Q_2 respectively, so that we have

$$Q = Q_1 + Q_2$$

The generators have simple linear cost functions

$$C_1(Q_1) = c_1 \cdot Q_1$$
$$C_2(Q_2) = c_2 \cdot Q_2$$

where we assume $c_1 < c_2$. The dispatch must be positive, but below the capacity:

$$0 \leq Q_1 \leq \hat{Q}_1$$
$$0 \leq Q_2 \leq \hat{Q}_2$$

Several generators with fixed demand

What is the most efficient dispatch of the two generators?

We want to maximise the total economic welfare, which is the sum of generator and consumer surplus. For this example, we have no control of the consumer side, so we must seek to maximise the generator surplus. This is equivalent to minimising the costs. We can write an optimisation problem with objective function:

$$\max_{Q_1, Q_2} f(Q_1, Q_2) = -c_1 \cdot Q_1 - c_2 \cdot Q_2$$

subject to one equality and four inequality constraints:

$$Q - Q_1 - Q_2 = 0$$

$$0 \leq Q_1 \leq \hat{Q}_1$$

$$0 \leq Q_2 \leq \hat{Q}_2$$

Several generators with fixed demand: Example 1

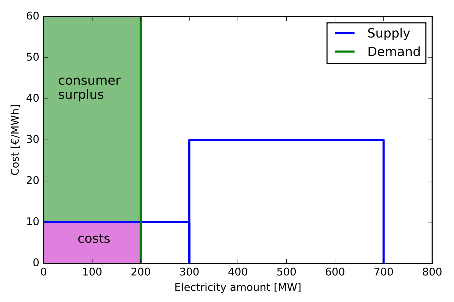
Suppose $c_1 = 10$ €/MWh, $c_2 = 30$ €/MWh, $Q = 200$ MW, $\hat{Q}_1 = 300$ MW, $\hat{Q}_2 = 400$ MW.

What is the optimal power plant dispatch, i.e. what values of Q_1, Q_2 maximise efficiency?

Answer is clear: supply all of the demand with the cheapest generator 1.

The optimal dispatch is $Q_1^* = 200 < \hat{Q}_1$ and $Q_2^* = 0 < \hat{Q}_2$.

In this example the inequalities for generator 1 are non-binding.



Equality Constraints and Shadow Prices

We have an equality constraint:

$$Q - Q_1 - Q_2 = 0$$

We can ask: what is the change in the objective function, i.e. the increase in total cost, if we increase the load Q by one unit?

In this case, the extra load would be supplied by generator 1, which still has capacity, so the shadow price of the balance constraint is $\lambda^* = c_1$.

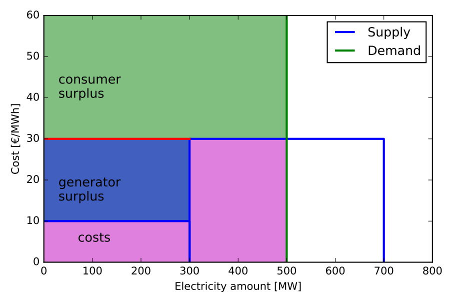
Several generators with fixed demand: Example 2

Suppose now we raise the load to $Q = 500$ MW. What values of Q_1, Q_2 maximise efficiency? Answer is clear: max out cheapest generator 1, then supply the remaining demand with the more expensive generator 2.

The optimal dispatch is $Q_1^* = 300 = \hat{Q}_1$ and $Q_2^* = 200 < \hat{Q}_2$.

The balance constraint here has a shadow price of $\lambda^* = c_2$.

In this example the inequality for generator 1 is binding but the inequality for generator 2 is non-binding.



Binding Inequality Constraints and Shadow Prices

If we associate shadow prices μ_i with the four generation inequality constraints

$$\begin{aligned}
Q_1 \leq \hat{Q}_1 \qquad &\leftrightarrow \qquad \mu_1 \\
Q_2 \leq \hat{Q}_2 \qquad &\leftrightarrow \qquad \mu_2 \\
-Q_1 \leq \hat{0} \qquad &\leftrightarrow \qquad \mu_3 \\
-Q_2 \leq \hat{0} \qquad &\leftrightarrow \qquad \mu_4
\end{aligned}$$

Then we can ask: what is the change in the objective function if we relax the first constraint, i.e. allow one more unit of capacity $\hat{Q}_1$ for generator 1.

We substitute generation from generator 2 with generation from generator 1, thus increasing total welfare by

$$\mu_1^* = c_2 - c_1$$

Of course this would have to be balanced against the capital costs of the addition of capacity to generator 1.

General constrained optimisation theory: Lagrangians and Karush-Kuhn-Tucker conditions

Optimisation problem

We have an *objective function* $f : \mathbb{R}^k \to \mathbb{R}$

$$\max_x f(x)$$

$[x = (x_1, \ldots x_k)]$ subject to some constraints within $\mathbb{R}^k$:

$$\begin{array}{cccc} g_i(x) = c_i & \leftrightarrow & \lambda_i & i = 1, \ldots n \\ h_j(x) \leq d_j & \leftrightarrow & \mu_j & j = 1, \ldots m \end{array}$$

λ_i and μ_j are the KKT 'Lagrange' multipliers we introduce for each constraint equation; their meaning and interpretation will be explained in the next slide.

Lagrangian

We now study the *Lagrangian function*

$$\mathcal{L}(x, \lambda, \mu) = f(x) - \sum_i \lambda_i \left[g_i(x) - c_i\right] - \sum_j \mu_j \left[h_j(x) - d_j\right]$$

We've built this function using the variables λ_i and μ_j to better understand the optimal solution of $f(x)$ given the constraints.

The optima of $\mathcal{L}(x, \lambda, \mu)$ tell us important information about the optima of $f(x)$ given the constraints.

We can already see that if $\frac{\partial \mathcal{L}}{\partial \lambda_i} = 0$ then the equality constraint $g_i(x) = c$ will be satisfied.

[Beware: $\pm$ signs appear differently in literature, but have been chosen here such that $\lambda_i = \frac{\partial \mathcal{L}}{\partial c_i}$ and $\mu_j = \frac{\partial \mathcal{L}}{\partial d_j}$.]

KKT conditions

The Karush-Kuhn-Tucker (KKT) conditions are necessary conditions that an optimal solution x^*, μ^*, λ^* always satisfies (up to some regularity conditions):

1. Stationarity: For $l = 1, \ldots k$
$$\frac{\partial \mathcal{L}}{\partial x_l} = \frac{\partial f}{\partial x_l} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial x_l} - \sum_j \mu_j^* \frac{\partial h_j}{\partial x_l} = 0$$
2. Primal feasibility:
$$g_i(x^*) = c_i$$
$$h_j(x^*) \leq d_j$$
3. Dual feasibility: $\mu_j^* \geq 0$
4. Complementary slackness: $\mu_j^*(h_j(x^*) - d_j) = 0$

Complementarity slackness for inequality constraints

We have for each inequality constraint

$$\mu_j^* \geq 0$$
$$\mu_j^*(h_j(x^*) - d_j) = 0$$

So either the inequality constraint is binding

$$h_j(x^*) = d_j$$

and we have $\mu_j^* \geq 0$.

Or the inequality constraint is NOT binding

$$h_j(x^*) < d_j$$

and we therefore MUST have $\mu_j^* = 0$.

If the inequality constraint is non-binding, we can remove it from the optimisation problem, since it has no effect on the optimal solution.

Nota Bene

1. The KKT conditions are only sufficient for optimality of the solution under certain conditions, e.g. linearity of the problem.

2. Since at the optimal solution we have $g_i(x^*) = c_i$ for equality constraints and $\mu_j^*(h_j(x^*) - d_j) = 0$, we have

$$\mathcal{L}(x^*, \lambda^*, \mu^*) = f(x^*)$$

KKT and Welfare Maximisation 1/2

Apply KKT now to maximisation of total economic welfare:

$$\max_{\{Q_i^B\},\{Q_i^S\}} f(\{Q_i^B\},\{Q_i^S\}) = \left[\sum_i U_i(Q_i^B) - \sum_i C_i(Q_i^S)\right]$$

subject to the balance constraint:

$$g(\{Q_i^B\},\{Q_i^S\}) = \sum_i Q_i^B - \sum_i Q_i^S = 0 \qquad \leftrightarrow \qquad \lambda$$

and any other constraints (e.g. limits on generator capacity, etc.).

Our optimisation variables are $\{x\} = \{Q_i^B\} \cup \{Q_i^S\}$.

We get from stationarity:

$$0 = \frac{\partial f}{\partial Q_i^B} - \sum_i \lambda^* \frac{\partial g}{\partial Q_i^B} = U_i'(Q_i^B) - \lambda^* = 0$$

$$0 = \frac{\partial f}{\partial Q_i^S} - \sum_i \lambda^* \frac{\partial g}{\partial Q_i^S} = -C_i'(Q_i^S) + \lambda^* = 0$$

KKT and Welfare Maximisation 2/2

So at the optimal point of maximal total economic welfare we get the same result as if everyone maximises their own welfare separately:

$$U_i'(Q_i^B) = \lambda^*$$
$$C_i'(Q_i^S) = \lambda^*$$

This is the CENTRAL result of microeconomics.

If we have further inequality constraints that are binding, then these equations will receive additions with $\mu_i^* > 0$, as we will see in the next examples...

Application to 2-generator example with fixed demand

Our optimisation variables are $\{x\} = \{Q_1, Q_2\}$ with objective function

$$\max_{Q_1,Q_2} f(Q_1, Q_2) = -c_1 \cdot Q_1 - c_2 \cdot Q_2$$

subject to one equality and four inequality constraints:

$$\begin{aligned} Q - Q_1 - Q_2 = 0 \qquad &\leftrightarrow \qquad \lambda \\ Q_1 \leq \hat{Q}_1 \qquad &\leftrightarrow \qquad \mu_1 \\ Q_2 \leq \hat{Q}_2 \qquad &\leftrightarrow \qquad \mu_2 \\ -Q_1 \leq \hat{0} \qquad &\leftrightarrow \qquad \mu_3 \\ -Q_2 \leq \hat{0} \qquad &\leftrightarrow \qquad \mu_4 \end{aligned}$$

KKT for example

Stationarity gives us:

$$-c_1 + \lambda^* - \mu_1^* + \mu_3^* = 0$$
$$-c_2 + \lambda^* - \mu_2^* + \mu_4^* = 0$$

Dual feasibility and complementary slackness give us:

$$\begin{aligned} \mu_i^* &\geq 0 && \forall i \in 1, 2, 3, 4 \\ \mu_i^*(Q_i^* - \hat{Q}_i) &= 0 && \forall i \in 1, 2 \\ \mu_i^*(-Q_i^*) &= 0 && \forall i \in 3, 4 \end{aligned}$$

Application to Example 1

Here we have no binding constraints for generator 1 at the optimal point, $0 < Q_1^* < \hat{Q}_1$, and generator 2 is at the lower boundary $Q_2^* = 0$.

Thus we have $\mu_1^* = \mu_2^* = \mu_3^* = 0$ and $\mu_4^* \neq 0$.

We get from stationarity

$$-c_1 + \lambda^* = 0$$
$$-c_2 + \lambda^* + \mu_4^* = 0$$

Thus the non-zero Lagrange multipiers are:

$$\lambda^* = c_1$$
$$\mu_4^* = c_2 - c_1$$

Application to Example 2

Here we have one binding constraint for generator 1 at the optimal point, $Q_1^* = \hat{Q}_1$, and generator 2 has no binding constraints $0 < Q_2^* < \hat{Q}_2$.

Thus we have $\mu_1^* \neq 0$, $\mu_2^* = \mu_3^* = \mu_4^* = 0$.

We get from stationarity

$$\begin{aligned} -c_1 + \lambda^* - \mu_1^* &= 0 \\ -c_2 + \lambda^* &= 0 \end{aligned}$$

Thus the non-zero Lagrange multipiers are:

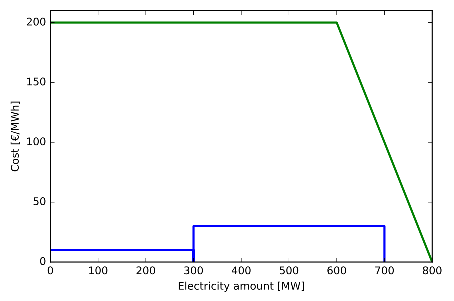
$$\begin{aligned} \lambda^* &= c_2 \\ \mu_1^* &= c_2 - c_1 \end{aligned}$$

Extension of Example for Demand Higher than Generation

In the previous examples there was always one generator setting the price, getting revenue at their marginal cost. How can such generators every recoup their capital costs?

Suppose now we promote the demand Q to an optimisation variable with utility

$$U(Q) = \begin{cases} 200Q & \text{for } Q \leq 600 \\ -180000 + 800Q - \frac{1}{2}Q^2 & \text{for } 600 \leq Q \leq 800 \end{cases}$$



Application to 2-generator example with fixed demand

Our optimisation variables are $\{x\} = \{Q, Q_1, Q_2\}$ with objective function

$$\max_{Q,Q_1,Q_2} f(Q, Q_1, Q_2) = U(Q) - c_1 \cdot Q_1 - c_2 \cdot Q_2$$

subject to one equality and four inequality constraints:

$$\begin{aligned}
Q - Q_1 - Q_2 = 0 \quad &\leftrightarrow \quad \lambda \\
Q_1 \leq \hat{Q}_1 \quad &\leftrightarrow \quad \mu_1 \\
Q_2 \leq \hat{Q}_2 \quad &\leftrightarrow \quad \mu_2 \\
-Q_1 \leq \hat{0} \quad &\leftrightarrow \quad \mu_3 \\
-Q_2 \leq \hat{0} \quad &\leftrightarrow \quad \mu_4
\end{aligned}$$

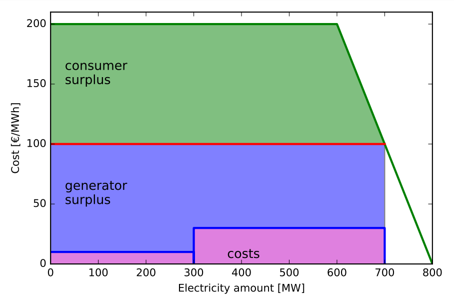
Extension of Example for Demand Higher than Generation

We max out both generators $Q_1^* = 300, Q_2^* = 400, Q^* = 700$.

From KKT we get $\mu_3^* = \mu_4^* = 0$ because the generators are both at their *upper limits*.

KKT stationarity:

$$
\begin{aligned}
U'(Q^*) - \lambda^* &= 0 & &\Rightarrow \lambda^* = 100 \\
-c_1 + \lambda^* - \mu_1^* &= 0 & &\Rightarrow \mu_1^* = \lambda^* - c_1 = 90 \\
-c_2 + \lambda^* - \mu_2^* &= 0 & &\Rightarrow \mu_2^* = \lambda^* - c_2 = 70
\end{aligned}
$$



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