Energy Systems, Summer Semester 2025

Lecture 5: Power Flow

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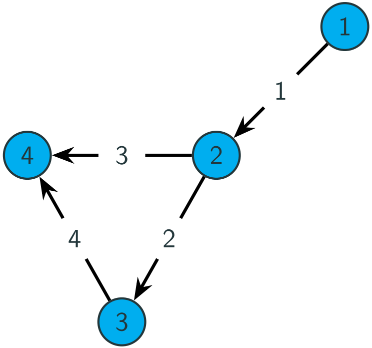
Graph Theory

The goal of power flow analysis

The goal of a power/load flow analysis is to find the flows in the lines of a network given a power injection pattern at the nodes.
I.e. given power injection at the nodes

$$\mathbf{P}_i = \begin{pmatrix} 50 \\ 50 \\ 0 \\ -100 \end{pmatrix}$$

what are the flows in lines 1-4?



Definition of a network

Our definition (Newman): A **network** (graph) is a collection of **vertices** (nodes) joined by **edges** (links).

More precise definition (Bollobàs): An undirected graph G is an ordered pair of disjoint sets (V, E) such that E (the edges) is a subset of the set $V^{(2)}$ of unordered pairs of V (the vertices).

Edge list representation

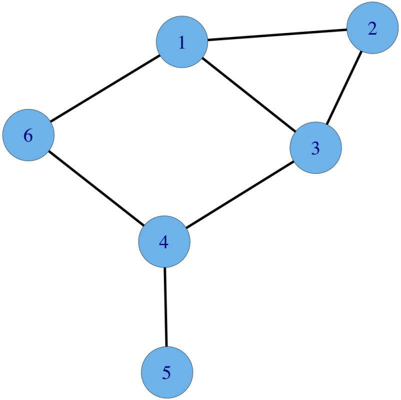
- Vertices:
  1,2,3,4,5,6

- Edges:
  (1,2), (1,3), (1,6), (2,3), (3,4), (4,5), (4,6)

Definition from graph theory:

- $N = 6$ vertices: **order** of the graph

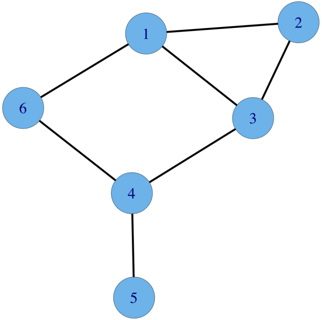
- $L = 7$ edges: **size** of the graph



Adjacency matrix A

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge between vertices i and j} \\ 0 & \text{otherwise.} \end{cases}$$

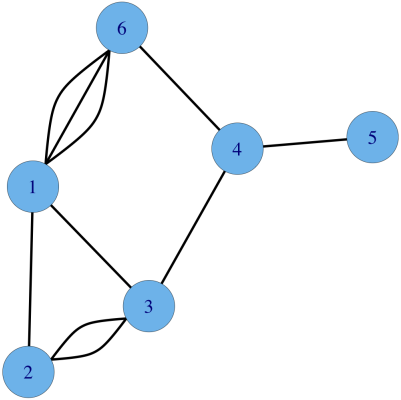
$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



- Diagonal elements are zero.
- Symmetric matrix for an undirected graph.
- If there are N vertices, it's an $N \times N$ matrix.

There can be more than one edge between a pair of vertices.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 3 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

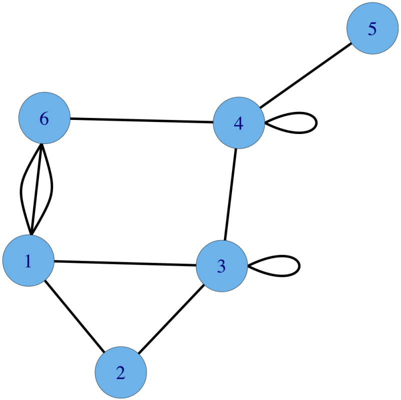


Self-edges

There can be **self-edges** (also called self-loops).

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 3 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

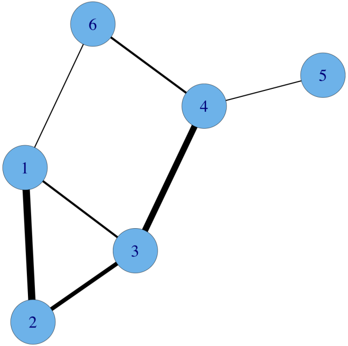
- Diagonal elements can be non-zero:
  Definition: $A_{ii} = 2$ for one self-edge.



Weighted networks

We can assign a **weight** or **strength** assigned to each edge.

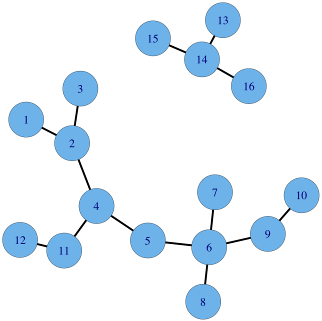
$$\mathbf{A} = \begin{pmatrix} 0 & 1.4 & 0.4 & 0 & 0 & 0.8 \\ 1.4 & 0 & 1.2 & 0 & 0 & 0 \\ 0.4 & 1.2 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0.8 & 0 & 0 & 0.4 & 0 & 0 \end{pmatrix}$$



Weights can be both positive or negative.

Components of networks

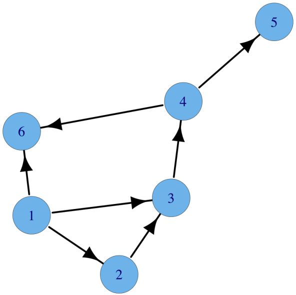
- Subgroups of vertices with no connections between the respective groups
- **Disconnected** network
- Subgroups: **components**
- Adjacency matrix: Block-diagonal form



A graph is **directed** if each edge is pointing from one vertex to another (**directed edge**).

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge from } j \text{ to } i \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



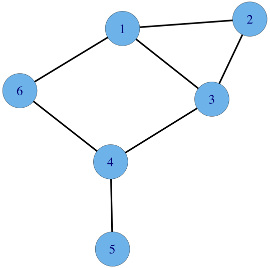
In general the adjacency matrix of a directed network is asymmetric.

- The **degree** k_i of a vertex i is defined as the number of edges connected to i.
- Average degree of the network: $\langle k \rangle$.

In terms of the adjacency matrix $\mathbf{A}$:

$$k_i = \sum_{j=1}^{n} A_{ij} \quad , \quad \langle k \rangle = \frac{1}{n} \sum_{i} k_i = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \ .$$

$$k_5 = 1$$
$$k_2 = k_6 = 2$$
$$k_1 = k_3 = k_4 = 3$$
$$\langle k \rangle = 2.33$$



Examples

NETWORK	NODES	LINKS	DIRECTED UNDIRECTED	N	L	⟨k⟩
Internet	Routers	Internet connections	Undirected	192,244	609,066	6.34
WWW	Webpages	Links	Directed	325,729	1,497,134	4.60
Power Grid	Power plants, transformers	Cables	Undirected	4,941	6,594	2.67
Mobile Phone Calls	Subscribers	Calls	Directed	36,595	91,826	2.51
Email	Email addresses	Emails	Directed	57,194	103,731	1.81
Science Collaboration	Scientists	Co-authorship	Undirected	23,133	93,439	8.08
Actor Network	Actors	Co-acting	Undirected	702,388	29,397,908	83.71
Citation Network	Paper	Citations	Directed	449,673	4,689,479	10.43
E. Coli Metabolism	Metabolites	Chemical reactions	Directed	1,039	5,802	5.58
Protein Interactions	Proteins	Binding interactions	Undirected	2,018	2,930	2.90

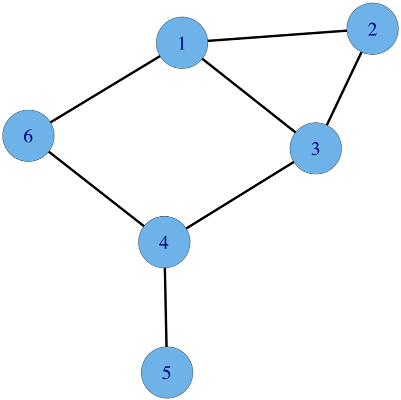
Source: Network Science (2015) by Albert-László Barabási

Degree matrix D

$$D_{ij} = \begin{cases} k_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

It's an $N \times N$ matrix again:

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$



Laplacian L

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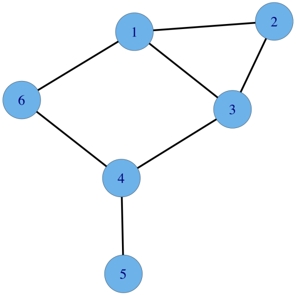
The **Laplacian matrix** is an $N \times N$ matrix defined for an undirected graph by

$$\mathbf{L} = \mathbf{D} - \mathbf{A}$$

$$\mathbf{L} = \begin{pmatrix} 3 & -1 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$



- $\mathbf{L}$ inherits symmetry from $\mathbf{D}$ and $\mathbf{A}$ for the undirected graph.
- The columns (and rows) sum to zero, since for each node, the degree equals the number of adjacent nodes.
- For a set of connected nodes I, $\sum_{i \in I} L_{ij} = 0 \ \forall j$.

Eigenvalues and eigenvectors of Laplacian L

The number of eigenvectors with zero eigenvalues equals the number of connected components.

For our connected graph, the single zero eigenvector is $(1, 1, \ldots 1)$:

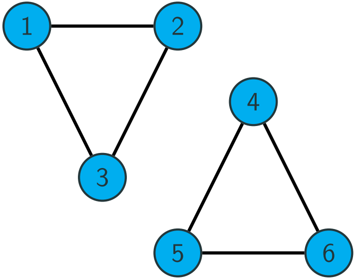
$$\mathbf{L}\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}3 & -1 & -1 & 0 & 0 & -1\\ -1 & 2 & -1 & 0 & 0 & 0\\ -1 & -1 & 3 & -1 & 0 & 0\\ 0 & 0 & -1 & 3 & -1 & -1\\ 0 & 0 & 0 & -1 & 1 & 0\\ -1 & 0 & 0 & -1 & 0 & 2\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}$$

Multiplying this eigenvector sums the rows, which gives zero.

The image of the matrix is made of differences across the nodes, so is $N - 1$ dimensional.

For a graph with two connected components, the Laplacian becomes block diagonal for the components, since there are no edges linking the components in the adjacency matrix:

$$\mathbf{L} = \begin{pmatrix} 2 & -1 & -1 & & & \\ -1 & 2 & -1 & & & \\ -1 & -1 & 2 & & & \\ & & & 2 & -1 & -1 \\ & & & -1 & 2 & -1 \\ & & & -1 & -1 & 2 \end{pmatrix}$$



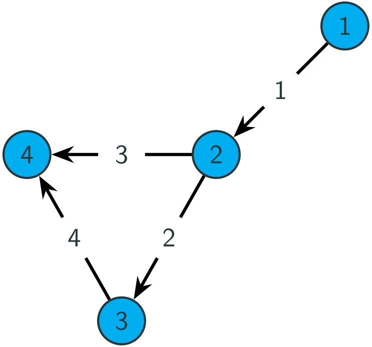
Verify that the two zero eigenvectors are $(1, 1, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 1, 1)$ corresponding to the connected components.

The incidence matrix

For a directed graph (every edge has an orientation) $G = (V, E)$ with N nodes and L edges, the node-edge **incidence matrix** $K \in \mathbb{R}^{N \times L}$ has components

$$K_{i\ell} = \begin{cases} 1 & \text{if edge } \ell \text{ starts at node } i \\ -1 & \text{if edge } \ell \text{ ends at node } i \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{K} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$



Incidence matrix properties

The incidence matrix has several important properties.

First, for a given edge ℓ, the corresponding column sums to zero $\sum_i K_{i\ell} = 0$, since every edge starts at some node (+1) and ends at some node (-1).

The row corresponding to each node i tells you which edges start there (+1) and which edges end there (-1).

It is related to the Laplacian matrix by

$$L = KK^t$$

Check the definitions agree:

$$L_{ij} = \sum_{\ell} K_{i\ell} K_{j\ell}$$

for $i = j$ and $i \neq j$.

NB: K is defined for a directed graph, but L for the undirected version. The information on the direction of the edges is lost in the formula $L = KK^t$.

Incidence matrix properties

Let's verify:

$$L_{ij} = \sum_{\ell} K_{i\ell} K_{j\ell}$$

For $i = j$ we get

$$L_{ii} = \sum_{\ell} (K_{i\ell})^2$$

The summands are only non-zero if the line ℓ is attached to i, so we get the degree

$$L_{ii} = k_i$$

Correct!

For $i \neq j$ we have

$$L_{ij} = \sum_{\ell} K_{i\ell} K_{j\ell}$$

If there is no line between i and j, then both sides are zero.

If there is a line between i and j one of $K_{i\ell}$ and $K_{j\ell}$ is $+1$, while the other is -1, so we get $L_{ij} = -1$. Correct!

The kernel of the incidence matrix

The kernel of $K_{i\ell}$, i.e. particular combinations of edges which are annihilated by K, has a very special meaning.

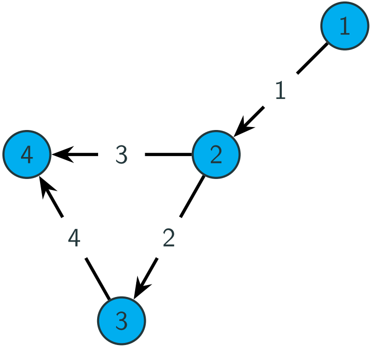
Consider the combination of edges $(0, 1, -1, 1)^t$

$$\mathbf{K}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

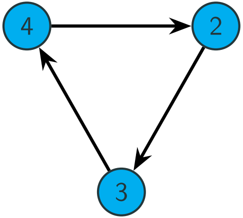
This corresponds to a **cycle** in the graph. A cycle is a path through the network that returns to its starting node. Each node in the cycle has an edge that ends there and an edge that starts there, so is annihilated by K.

A cycle

For our graph:



The combination of edges $(0, 1, -1, 1)^t$ corresponds to the cycle:



NB: The direction of edge 3 is reversed by the minus sign.

Cycle matrix

We can organise the cycles in a matrix $C_{\ell c}$, where c labels each cycle.

We have

$$KC = 0$$

by definition of C being in the kernel.

The image of K has dimension $N - 1$ (i.e. the rank of K) for a connected graph, since the space spanned by the columns of K can only reach differences between nodes and never then N-length vector $(1, 1, \ldots 1)^t$.

By the rank-nullity theorem for K we have

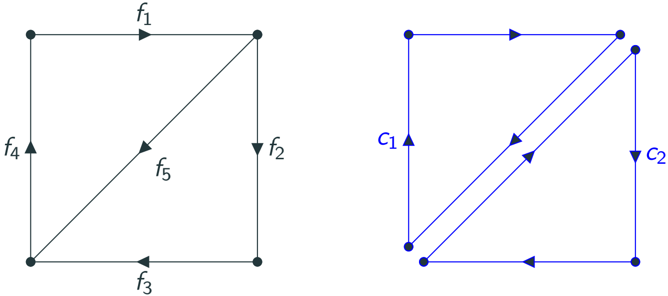
$$L = \dim \mathrm{im} K + \dim \ker K$$

so the number of cycles, i.e. the dimension of the kernel (nullity) of K is $L - N + 1$. If the connected graph has no cycles, i.e. it is a tree, then $L = N - 1$.

In our case $L = 4$, $N = 4$ so there is only 1 cycle

$$\mathbf{C} = (0, 1, -1, 1)^t$$

Independent basis of cycles



Two independent cycles:

$c_1 = f_1 + f_5 + f_4$

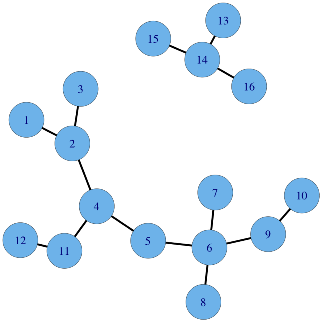
$c_2 = f_2 + f_3 + -f_5$

The outer cycle is not independent:

$c_3 = f_1 + f_2 + f_3 + f_4 = c_1 + c_2$

Trees

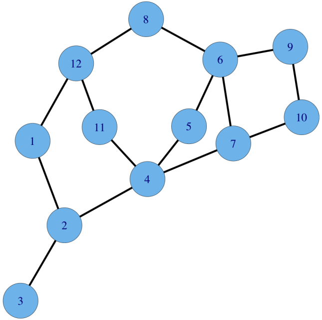
- Trees play an import role for random graph models.
- In a tree, there is exactly one path between any pair of vertices.
- A tree of N vertices always has exactly $N-1$ edges.
- Any connected network with N vertices and $N-1$ edges is a tree.
- Trees have **no cycles**.
- A collection of trees is called a **forest**.



A **planar network** is a network that can be drawn on a plane without having any edges cross.

Examples:

- Trees
- Road networks (approximately)
- Power grids (approximately)
- Shared borders between countries, etc.



Computing the Linear Power Flow

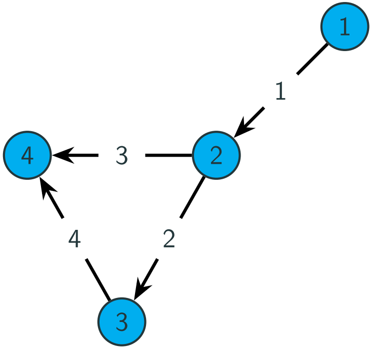
The goal of power flow analysis

The goal of a power/load flow analysis is to find the flows in the lines of a network given a power injection pattern at the nodes.
I.e. given power injection at the nodes

$$\mathbf{P}_i = \begin{pmatrix} 50 \\ 50 \\ 0 \\ -100 \end{pmatrix}$$

what are the flows in lines 1-4?

To find the flows, it is sufficient to know the **reactances** of the lines x_ℓ and the **voltages angles** θ_i at each node.



Framing the load flow problem

Suppose we have N nodes labelled by i, and L edges labelled by ℓ forming a directed graph G.

Suppose at each node we have a **power imbalance** p_i ($p_i > 0$ means its generating more than it consumes and $p_i < 0$ means it is consuming more than it).

Since we cannot create or destroy energy (and we're ignoring losses):

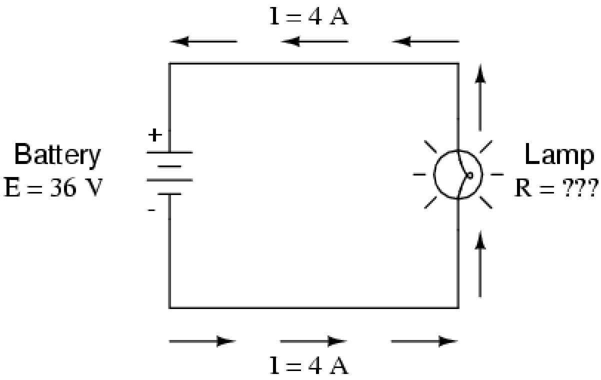
$$\sum_i p_i = 0$$

Question: How do the flows f_ℓ in the network relate to the nodal power imbalances?

Answer: According to the reactances (generalisation of resistance for oscillating voltage/current) and the corresponding voltages.

Ohm's Law: The potential difference (voltage) $V_1 - V_2$ across an ideal conductor is proportional to the current through it I. The constant of proportionality is called the **resistance**, R. Ohm's Law is thus:

$$V_1 - V_2 = I\,R$$



Analogy DC circuits to linear power flow

The equations for DC circuits and linear power flow in AC circuits are analogous:

$$I = \frac{V_i - V_j}{R} \qquad \leftrightarrow \qquad f_\ell = \frac{\theta_i - \theta_j}{x_\ell}$$

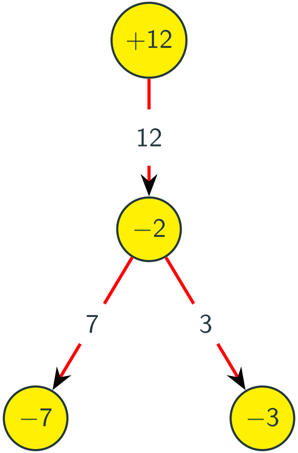
if we make the following identification:

Current flow I	$\leftrightarrow$	Active power flow f_ℓ
Potential/voltage V_i	$\leftrightarrow$	Voltage angle θ_i
Resistance R	$\leftrightarrow$	Reactance X

The simplifications that lead to the linear power flow will be explained later in the lecture.

Kirchhoff's Current Law (KCL)

KCL inforces energy conservation at each vertex (the power imbalance equals what goes out minus what comes in).



Kirchhoff's Current Law (KCL)

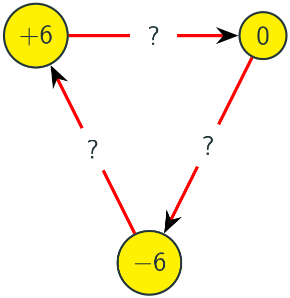
KCL says (in this linear setting) that the nodal power imbalance at node i is equal to the sum of direct flows arriving at the node. This can be expressed compactly with the incidence matrix

$$p_i = \sum_{\ell} K_{i\ell} f_\ell \qquad \forall i$$

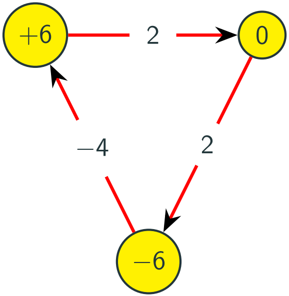
Only $N-1$ of these equations are independent for a connected network, since $\sum_i K_{i\ell} = 0$.

Kirchhoff's Voltage Law (KVL)

KCL isn't enough to determine the flow as soon as there are **closed cycles** in the network. For this we need Ohm's law in combination with KVL: voltage differences around each cycle add up to zero.



For equal reactances for each edge:



NB: For directed graph, sign determines direction of flow.

Kirchhoff's Voltage Law (KVL)

KVL says that the sum of voltage differences across edges for any closed cycle must add up to zero.

If the voltage angle at any node is given by θ_i then the voltage difference across edge ℓ is

$$\sum_i K_{i\ell}\theta_i$$

And Kirchhoff's law can be expressed using the cycle matrix encoding of independent cycles

$$\sum_\ell C_{\ell c} \sum_i K_{i\ell}\theta_i = 0 \qquad \forall c$$

[Automatic, since we already said KC = 0.]

Kirchhoff's Voltage Law (KVL)

Physics gives us the expression of the flow f_ℓ on each line ℓ with reactance x_ℓ in terms of the voltage angles at the nodes θ_i (a relative of $V = IR$)

$$f_\ell = \frac{\theta_i - \theta_j}{x_\ell} = \frac{1}{x_\ell} \sum_i K_{i\ell} \theta_i \tag{1}$$

[NB: This restricts the L variables f_ℓ to depend only on the N voltage angles θ_i. Since the flow doesn't change under a constant shift $\theta_i \to \theta_i + c$, we can choose a **slack** or **reference node** such that $\theta_1 = 0$, so there are only $N - 1$ independent variables.]

KVL now becomes $L - N + 1$ binding constraints on the line flows f_ℓ

$$\sum_\ell C_{\ell c} x_\ell f_\ell = 0 \qquad \forall c \tag{2}$$

[NB: Equations (1) and (2) are equivalent and both restrict our L variables f_ℓ to an $N - 1$ dimensional subspace.]

Solving the equations via the line flows

Now we have $N - 1$ equations for the flows f_ℓ from KCL:

$$p_i = \sum_\ell K_{i\ell} f_\ell \qquad \forall i \in \{1, \ldots N-1\}$$

and $L - N + 1$ equations from KVL:

$$\sum_\ell C_{\ell c} x_\ell f_\ell = 0 \qquad \forall c \in \{1, \ldots L - N + 1\}$$

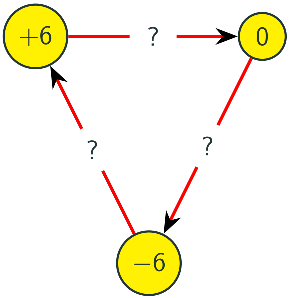
So L independent linear equations for L variables f_ℓ.

Can solve with e.g. LU decomposition using specialised sparse solvers, with polynomial complexity in L. (For dense matrices complexity $O(L^a)$ where $2 < a < 3$.)

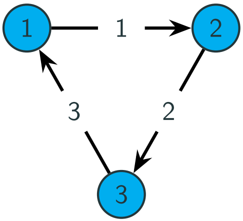
This formulation is useful for the optimisation later, but we can solve a smaller dimensional linear system with $N - 1$ variables using the voltage angles.

Solving 3-node example

In the 3-node example energy conservation at each vertex (Kirchhoff's Current Law, KCL) was not enough information to solve the power flow, since there are multiple paths in the network. Assume equal reactances $x_\ell = x$ on each edge.

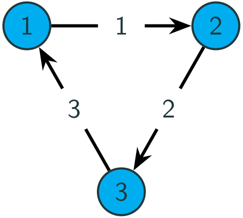


Formalise by labelling the nodes and edges:



We have $p_i = (6, 0, -6)$. (Check $\sum_i p_i = 0$.)
Goal is to find f_ℓ for $\ell = 1, 2, 3$.

Solving 3-node example: Kirchhoff's Current Law (KCL)



Kirchhoff's Current Law gives us:

$$p_i = \sum_\ell K_{i\ell} f_\ell \qquad \forall i$$

The incidence matrix K is given by:

$$\mathbf{K}_{i\ell} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

So we get:

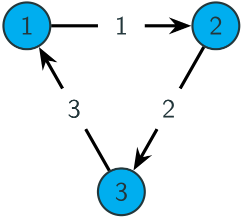
$$p_1 = 6 = f_1 - f_3$$
$$p_2 = 0 = f_2 - f_1$$
$$p_3 = -6 = f_3 - f_2$$

Sum of KCL equations is always zero, so reduce to $N - 1 = 2$ independent equations:

$$6 = f_1 - f_3$$
$$0 = f_2 - f_1$$

Not enough information to solve!

Need more information from KVL and reactances.



One formulation of Kirchhoff's Voltage Law gives us $L - N + 1$ equations for cycles:

$$\sum_\ell C_{\ell c} x_\ell f_\ell = 0 \qquad \forall c$$

The cycle matrix C is given by:

$$\mathbf{C}_{\ell c} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For equal reactances $x_\ell = x$ we get:

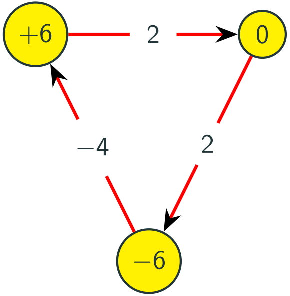
$$\sum_\ell C_{\ell 1} x_\ell f_\ell = x(f_1 + f_2 + f_3) = 0$$

Together with KCL equations we now have 3 independent equations for 3 unknowns. Solve:

$$f_1 = 2$$
$$f_2 = 2$$
$$f_3 = -4$$

Solving 3-node example: Solution

Solution:



Along 2-edge path reactance is double the 1-edge path, so half as much power flows along the 2-edge path as the 1-edge path.

NB: For directed graph, sign determines direction of flow.

Solving the equations via the voltage angles

If we combine

$$f_\ell = \frac{1}{x_\ell} \sum_i K_{i\ell} \theta_i \tag{3}$$

with Kirchhoff's Current Law we get

$$p_i = \sum_\ell K_{i\ell} f_\ell = \sum_\ell K_{i\ell} \frac{1}{x_\ell} \sum_j K_{j\ell} \theta_j$$

This is a **weighted Laplacian**. If we write $B_{k\ell}$ for the diagonal matrix with $B_{\ell\ell} = \frac{1}{x_\ell}$ then

$$L = KBK^t$$

and we get a **discrete Poisson equation** for the θ_i sourced by the p_i

$$p_i = \sum_j L_{ij} \theta_j$$

This is a set of $N-1$ sparse linear equations for the θ_j ($N-1$ since $\sum_i L_{ij} = 0$). We can solve this for the θ_i and then find the flows using equation (3). Polynomial complexity in N.

Solving the equations via the PTDF

If we are repeating the calculation for a fixed network multiple times with different power injections, it can make sense to do the full matrix inversion.

Given p_i at every node, we want to find the flows f_ℓ. We have the equations

$$p_i = \sum_j L_{ij}\theta_j$$

$$f_\ell = \frac{1}{x_\ell}\sum_i K_{i\ell}\theta_i$$

Basic idea: invert L to get θ_i in terms of p_i

$$\theta_i = \sum_k (L^{-1})_{ik} p_k$$

then insert to get the flows as a linear function of the power injections p_i

$$f_\ell = \frac{1}{x_\ell}\sum_{i,k} K_{i\ell}(L^{-1})_{ik} p_k = \sum_k \mathrm{PTDF}_{\ell k} p_k$$

called the **Power Transfer Distribution Factors** (PTDF).

Inverting Laplacian L

There is one small catch: L is **not invertible** since it has (for a connected network) one zero eigenvalue, with eigenvector $(1, 1, \dots 1)$, since by construction $\sum_j L_{ij} = 0$.

This is related to a gauge freedom to add a constant to all voltage angles

$$\theta_i \to \theta_i + c$$

which does not affect physical quantities:

$$p_i = \sum_j L_{ij}(\theta_j + c) = \sum_j L_{ij}(\theta_j)$$

$$f_\ell = \frac{1}{x_\ell} \sum_i K_{i\ell}(\theta_i + c) = \frac{1}{x_\ell} \sum_i K_{i\ell}(\theta_i)$$

Typically choose a **slack** or **reference node** such that $\theta_1 = 0$.

Inverting Laplacian L

Two solutions:

1. Since $\theta_1 = 0$ and p_1 is not independent of the other power injections ($\sum_{i=1}^N p_i = 0$ implies $p_1 = -\sum_{i=2}^N p_i$), we can ignore these elements and invert the lower-right $(N-1) \times (N-1)$ part of L (which doesn't have zero eigenvalues) to find the remaining $\{\theta_i\}_{i=2,\dots N}$ in terms of the $\{p_i\}_{i=2,\dots N}$.
2. Use the Moore-Penrose pseudo-inverse.

Write L in terms of its basis of orthonormal eigenvectors e_i^n ($\sum_j L_{ij} e_j^n = \lambda_n e_i^n$, $\sum_i e_i^n e_i^n = 1$ and $\sum_i e_i^n e_i^m = 0$ if $n \neq m$):

$$L_{ij} = \sum_n \lambda_n e_i^n e_j^n$$

then the Moore-Penrose pseudo-inverse is:

$$L_{ij}^\dagger = \sum_{n|\lambda_n \neq 0} \frac{1}{\lambda_n} e_i^n e_j^n$$

Check the Moore-Penrose pseudo-inverse

Let's check the Moore-Penrose pseudo-inverse really gives us an inverse:

$$\begin{aligned}
\sum_j L_{ij} L^\dagger_{jk} &= \sum_j \sum_n \lambda_n e_i^n e_j^n \sum_{m|\lambda_m \neq 0} \frac{1}{\lambda_m} e_j^m e_k^m \\
&= \sum_n \lambda_n e_i^n \sum_{m|\lambda_m \neq 0} \frac{1}{\lambda_m} e_k^m \sum_j e_j^n e_j^m \\
&= \sum_{m|\lambda_m \neq 0} \frac{\lambda_m}{\lambda_m} e_i^m e_k^m \\
&= \sum_{m|\lambda_m \neq 0} e_i^m e_k^m
\end{aligned}$$

From line 2 to 3 we use the orthogonality of the eigenvectors.

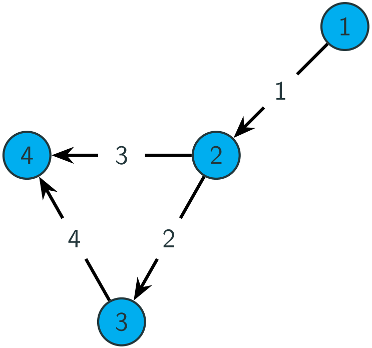
This is almost the identity. It has eigenvalues 1 for each eigenvector e_k^n except for zero eigenvectors of L with $\lambda_n = 0$, which it annihilates.

4-node example

$$\mathbf{K}_{i\ell} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

$$\mathbf{L}_{ij} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

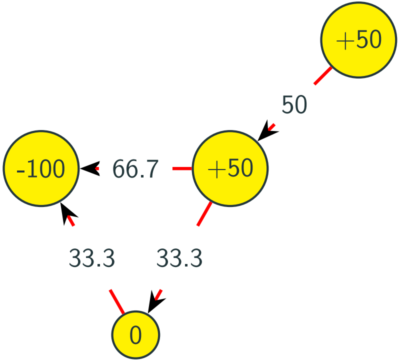
$$\mathbf{PTDF}_{\ell i} = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -2/3 & -1/3 \\ 0 & 0 & -1/3 & -2/3 \\ 0 & 0 & 1/3 & -1/3 \end{pmatrix}$$



4-node example

$$\sum_i \mathbf{PTDF}_{\ell i} p_i = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -2/3 & -1/3 \\ 0 & 0 & -1/3 & -2/3 \\ 0 & 0 & 1/3 & -1/3 \end{pmatrix} \begin{pmatrix} 50 \\ 50 \\ 0 \\ -100 \end{pmatrix}$$

$$= \begin{pmatrix} 50 \\ 33.3 \\ 66.7 \\ 33.3 \end{pmatrix}$$

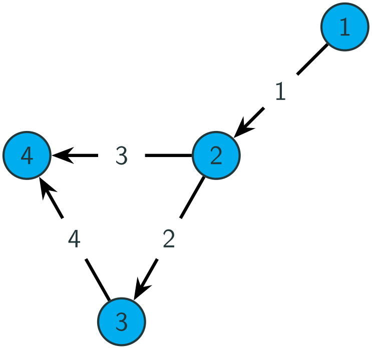


PTDF as sensitivity

Can also 'experimentally' determine the Power Transfer Distribution Factors (PTDF) by choosing a slack node (in this case node 1).

Each column (labelled by i) is then the resulting line flows if we have a simple power transfer from node i to the slack $p_i = 1$ and $p_1 = -1$.

$$\mathbf{PTDF}_{\ell i} = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -2/3 & -1/3 \\ 0 & 0 & -1/3 & -2/3 \\ 0 & 0 & 1/3 & -1/3 \end{pmatrix}$$



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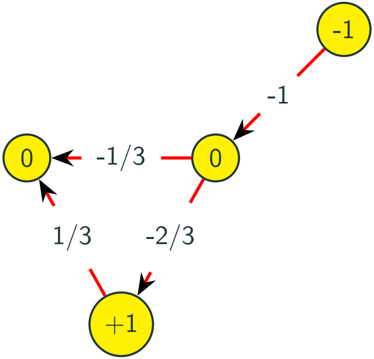
Focus on 3rd column of PTDF and look at power flow with $p_3 = +1$ and slack $p_1 = -1$. Coefficients determined by resulting flow:

$$\mathbf{PTDF}_{\ell 3} = \begin{pmatrix} -1 \\ -2/3 \\ -1/3 \\ 1/3 \end{pmatrix}$$



Consequences of Limiting Power Transfers

Line loading limits

You cannot pass infinite current through a transmission line.

As it warms, it sags, then it will become damaged and/or hit a building/tree and cause a short-circuit. For this reasons there are always **thermal limits** on current transfer. There may also be limits on the amount of power or current based on concerns about **voltage stability** or **general stability**.

Typically each line has a well-defined **line loading limit** on the amount of current or power that can flow through it:

$$|f_\ell| \leq F_\ell$$

where here F_ℓ is the maximum power capacity of the transmission line.

These limits prevent the transfer of renewable energy or other power sources.

Adjusting generator dispatch to avoid overloading

To avoid overloading the power lines, we must adjust our generator output (or the demand) so that the power imbalances do not overload the network.

We will now generalise and adjust our notation.

From lecture 3 we had for a single node:

$$-p_t = m_t - b_t + c_t = d_t - Ww_t - Ss_t - b_t + c_t = 0$$

where p_t was the nodal power balance, m_t was the mismatch (load d_t minus wind Ww_t and solar Ss_t), b_t was the backup power and c_t was curtailment.

We generalised this to multiple nodes labelled by i

$$-p_{i,t} = m_{i,t} - b_{i,t} + c_{i,t} = d_{i,t} - W_i w_{i,t} - S_i s_{i,t} - b_{i,t} + c_{i,t}$$

where now we don't enforce $p_{i,t} = 0$ but $\sum_i p_{i,t} = 0$ for all t.

Adjusting generator dispatch to avoid overloading

Now we write the dispatch of all generators at node i (wind, solar, backup) labelled by technology s as $g_{i,s,t}$ (i labels node, s technology and t time) so that we have a relation between load $d_{i,t}$, generation $g_{i,s,t}$ and network flows $f_{\ell,t}$

$$p_{i,t} = \sum_s g_{i,s,t} - d_{i,t} = \sum_\ell K_{i\ell} f_{\ell,t}$$

Where s runs over the wind, solar and backup capacity generators (e.g. hydro or natural gas) at the node.

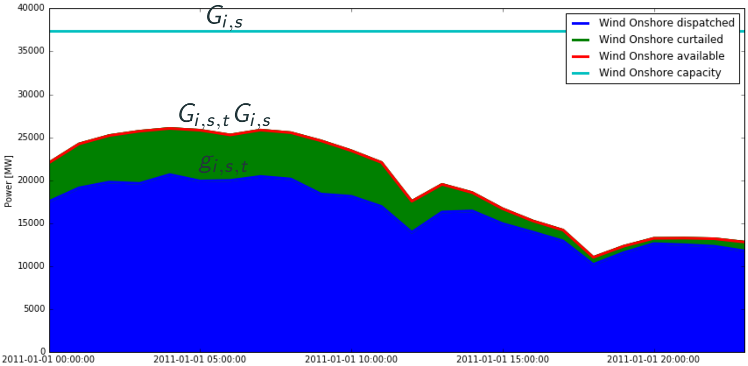
A dispatchable generator's $g_{i,s,t}$ output can be controlled within the limits of its power capacity $G_{i,s}$

$$0 \leq g_{i,s,t} \leq G_{i,s}$$

For a renewable generator we have time series of availability $0 \leq G_{i,s,t} \leq 1$ (the s_t and w_t before; W and S are the capacity $G_{i,s}$):

$$0 \leq g_{i,s,t} \leq G_{i,s,t} G_{i,s} \leq G_{i,s}$$

Curtailment corresponds to the case where $g_{i,s,t} < G_{i,s,t} G_{i,s}$:



Germany curtailment example

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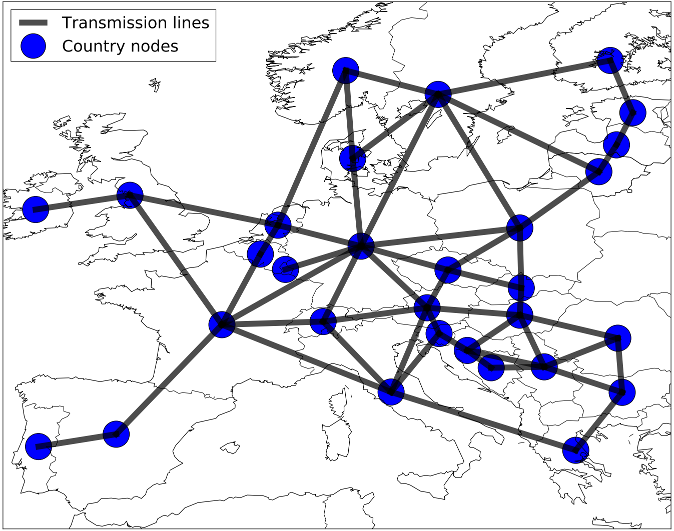
See **https://pypsa.readthedocs.io/en/latest/examples/scigrid-lopf-then-pf.html**.

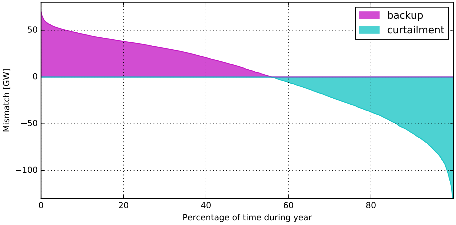
European transmission versus backup energy

Consider backup energy in a simplified European grid:

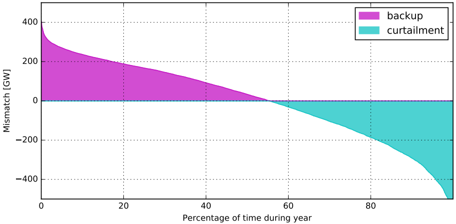


DE versus EU backup energy from last time

Germany needed backup generation for 31% of total load:



Europe needed backup generation for only 24% of the total load:



European transmission versus backup energy

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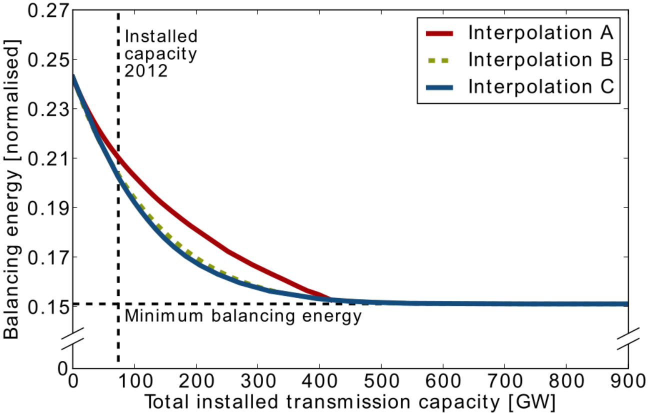
Transmission needs across a fully renewable European power system by Rodriguez, Becker, Andresen, Heide, Greiner, Renewable Energy, 2014

Cross-border capacities between countries are scaled up by interpolating either from today's capacities or from future optimal capacities, thereby reducing the need for balancing energy.



Full Power Flow Equations

We said we can (in the linear approximation) express the flow f_ℓ on each line in terms of the voltage angles θ_i at the nodes for a line ℓ with reactance x_ℓ as

$$f_\ell = \frac{\theta_i - \theta_j}{x_\ell} = \frac{1}{x_\ell} \sum_i K_{i\ell} \theta_i$$

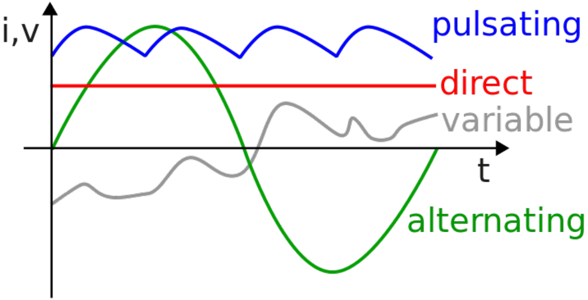
This is a relative of Ohm's Law in DC circuits, $I = \frac{V_1 - V_2}{R}$.

Now we explain the physics of where this comes from, and the linear approximation that leads to it.

This is also useful when we consider the synchronisation of oscillators later.

Alternating Current

The majority of electrical power, including what you get out of a wall plug, is transmitted as **Alternating Current (AC)**, i.e. both the voltage and current are sinusoidal waves.



[Some power is transmitted as **Direct Current (DC)** under bodies of water and indeed many electronic devices require DC (must convert AC to DC).]

Source: Wikipedia

Why alternating current?

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Battle of currents! Edison versus Westinghouse/Tesla in late 1880s, early 1890s, etc.

https://en.wikipedia.org/wiki/War_of_Currents

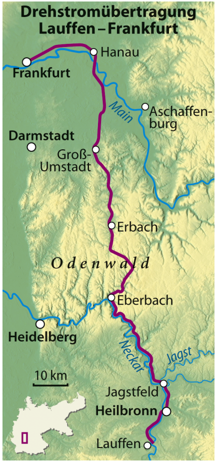
AC won, because it's easy to transform AC to a higher voltage, so you can transmit a given power $P = VI$ with a lower current and thus avoid the I^2R resistive losses in power lines.

Reason: $\frac{d}{dt}$ in $\mathcal{E} = \frac{d\Phi}{dt}$; use a solenoid to induce a **fluctuating** magnetic field in another solenoid with a different number of turns, giving different potential difference.

Frequency of 50 Hz is uniform across Europe (except for train-electricity, e.g. in Germany 16.7 Hz). 60 Hz in USA, western half of Japan, etc.

Frankfurt: Home of Long-Distance AC Transmission

First long-distance high-voltage alternating-current transmission in 1891 from hydroelectric plant in Lauffen to Frankfurt for the Elektrotechnische Ausstellung (176 km, 15 kV).



Source: Wikipedia

The voltage is usually written in terms of the **angular frequency** $\omega = 2\pi f$ (radians per second) rather than frequency f (Hertz) and the **Root-Mean-Squared (RMS)** voltage magnitude V_{rms}

$$V(t) = V_{\mathrm{peak}} \sin(\omega t) = \sqrt{2} V_{\mathrm{rms}} \sin(\omega t)$$

Similarly for the current we have

$$I(t) = I_{\mathrm{peak}} \sin(\omega t - \varphi) = \sqrt{2} I_{\mathrm{rms}} \sin(\omega t - \varphi)$$

Note that they are not necessarily in phase, $\varphi \neq 0$.

The RMS values are useful because then for the **average power** with $\varphi = 0$ we can forget factors of 2

$$\langle P(t) \rangle = \langle V(t) I(t) \rangle = 2 V_{\mathrm{rms}} I_{\mathrm{rms}} \langle \sin^2(\omega t) \rangle = V_{\mathrm{rms}} I_{\mathrm{rms}}$$

Resistive loads

For purely **resistive loads**, e.g. a kettle or an electric heater, we have

$$V(t) = RI(t)$$

and thus for a voltage of $V(t) = \sqrt{2}V_{\text{rms}}e^{j\omega t}$ (NB: for engineers $j = \sqrt{-1}$ to avoid confusion with the current i) we have

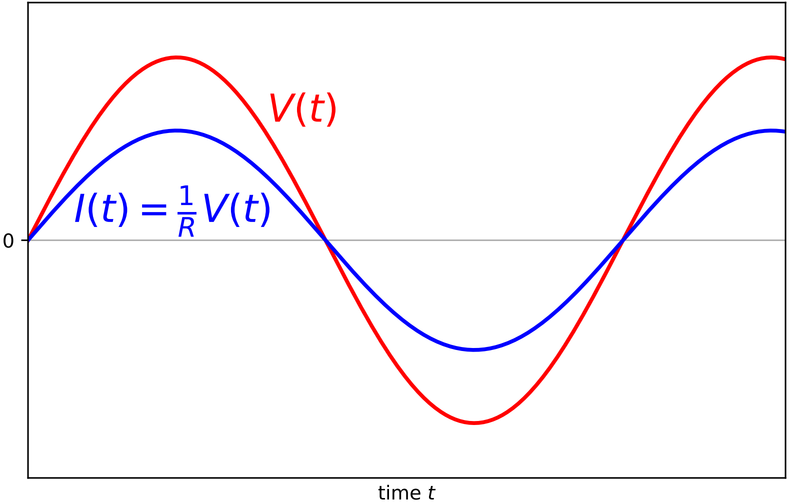
$$I(t) = \sqrt{2}\frac{V_{\text{rms}}}{R}e^{j\omega t} = \frac{1}{R}V(t)$$

or in terms of the RMS value and phase shift

$$I_{\text{rms}} = \frac{1}{R}V_{\text{rms}}$$

$$\varphi = 0$$

In terms of the waveforms, the current has no phase shift from the voltage.



Capacitive loads

For purely **capacitive loads** we have

$$I(t) = C\frac{dV(t)}{dt}$$

and thus for a voltage of $V(t) = \sqrt{2}V_{\mathrm{rms}}e^{j\omega t}$ we get

$$I(t) = \sqrt{2}j\omega C V_{\mathrm{rms}}e^{j\omega t} = j\omega C V(t)$$

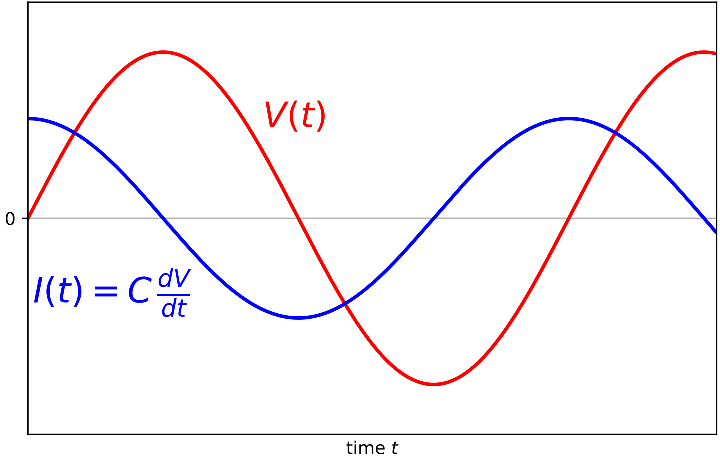
or in terms of the RMS value and phase shift

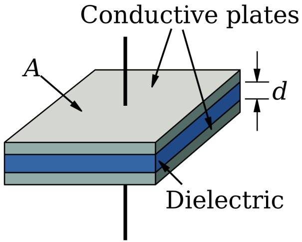
$$I_{\mathrm{rms}} = \omega C V_{\mathrm{rms}}$$
$$\varphi = -\frac{\pi}{2}$$

We write $X_C = \frac{1}{\omega C}$ for the **capacitive reactance**.

Capacitive loads

Current peaks before the voltage (it **leads** the voltage), since first charge must accumulate on the plates; once the charge is on the plates, the current drops to zero and the voltage peaks.





For purely **inductive loads**, e.g. a motor during start-up

$$V(t) = L\frac{dI(t)}{dt}$$

and thus for a voltage of $V(t) = \sqrt{2}V_{\mathrm{rms}}e^{j\omega t}$ we get

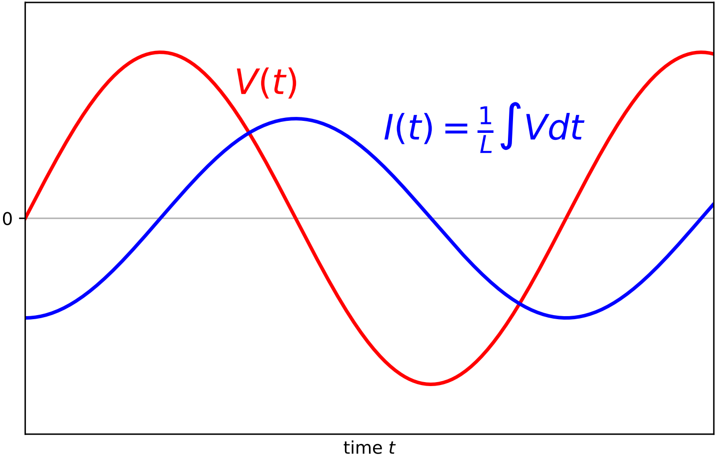
$$I(t) = \sqrt{2}\frac{V_{\mathrm{rms}}}{j\omega L}e^{j\omega t} = \frac{1}{j\omega L}V(t)$$

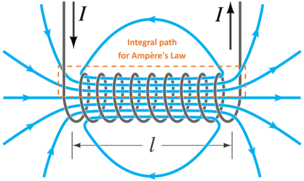
or in terms of the RMS value and phase shift

$$I_{\mathrm{rms}} = \frac{1}{\omega L}V_{\mathrm{rms}}$$
$$\varphi = \frac{\pi}{2}$$

We write $X_L = \omega L$ for the **inductive reactance**, in analogy to the resistance.

Now current peaks after the voltage (it **lags** the voltage), since the flow of current in the solenoid resists the changing voltage.





General loads

General loads will have a combination of resistive, capacitive and inductive parts. For an RLC circuit in series the voltage across the components is additive

$$V(t) = RI(t) + L\frac{dI(t)}{dt} + \frac{1}{C}\int_{-infty}^{t} I(\tau)d\tau$$

and therefore for a sinuisoidal voltage with angular frequency ω we get

$$V(t) = \left[R + j\omega L + \frac{1}{j\omega C}\right] I(t)$$

which leads us to define a general complex notion of resistance called **impedance**

$$Z = R + j\omega L + \frac{1}{j\omega C} = R + j(X_L - X_C) = R + jX$$

where X is the reactance $X = X_L - X_C$.

Impedances and admittances

Thus for a regular sinuisodal setup we have

$$V(t) = ZI(t)$$

where the complex **impedance** takes care both of the relation of the RMS values of the current and the voltage, and their phase difference. We can decompose Z into real resistance R and real reactance X

$$Z = R + jX$$

The inverse impedance, called the **admittance** is given by

$$Y = \frac{1}{Z}$$

so that

$$I(t) = YV(t)$$

We can also decompose this into real conductance G and real susceptance B

$$Y = G + jB$$

Simple transmission line

A simple model for a transmission line ℓ between nodes i and j is a resistance R in series with an (inductive) reactance X.

[Typical values are for a 380 kV overhead transmission line e.g. $R = 0.03$ Ohm/km and $X = 0.3$ Ohm/km.]

The voltage at each node (compared to ground) is given by $V_i(t) = \sqrt{2}V_i e^{j(\omega t+\theta_i)}$ where θ_i is the phase offset for each node and V_i is the RMS voltage magnitude.

Now the current in the transmission line is given by

$$I(t) = \frac{1}{R+jX}\left[V_j(t) - V_i(t)\right] = \frac{1}{R+jX}\sqrt{2}V_i e^{j(\omega t+\theta_i)}\left[\frac{V_j}{V_i}e^{j(\theta_j-\theta_i)} - 1\right]$$

Active versus reactive power

Now let's consider the power injection at the first node. This is simply the voltage there multiplied by the current in the transmission line.

It's convenient to eliminate the time-dependent part $e^{j\omega t}$ by multiplying the voltage with the complex conjugate of the current

$$S = P + jQ = \frac{1}{2} V(t) I^*(t)$$

For a resistive load with $V(t) = RI(t)$ this reproduces the **active power** P.

For loads where the $I(t)$ is not in phase with the voltage, we get a flow of **reactive power** Q.

$S = P + jQ$ is called the **apparent power**.

Linearisation: Assumption 1/3

Now if we consider the power injected at the first node we get

$$P_i + jQ_i = \frac{1}{R + jX} V_i^2 \left[\frac{V_j}{V_i} e^{j(\theta_i - \theta_j)} - 1 \right]$$

This is the full non-linear equation for the power flow. Now let's linearise by making some simplifying assumptions.

1. Assume the voltage magnitudes are the same everywhere in the network $V_i = V_j$

$$P_i + jQ_i = \frac{1}{R + jX} V_i^2 \left[e^{j(\theta_i - \theta_j)} - 1 \right]$$

This means **power flows primarily according to angle differences** in this approximation.

Linearisation: Assumption 2/3

2. Now assume that the voltage angle differences across the transmission line are small enough that $\sin(\theta_i - \theta_j) \sim (\theta_i - \theta_j)$

$$P_i + jQ_i = \frac{1}{R + jX} V_i^2 \left[e^{j(\theta_i - \theta_j)} - 1 \right]$$
$$\sim \frac{1}{R + jX} V_i^2 \left[j(\theta_i - \theta_j) \right]$$

This assumption is usually valid, since for stability reasons, we usually have in the transmission network $(\theta_i - \theta_j) \leq \frac{\pi}{6}$ (30 degrees).

Linearisation: Assumption 3/3

3. Finally we assume $R << X$ so that we can ignore the resistance R

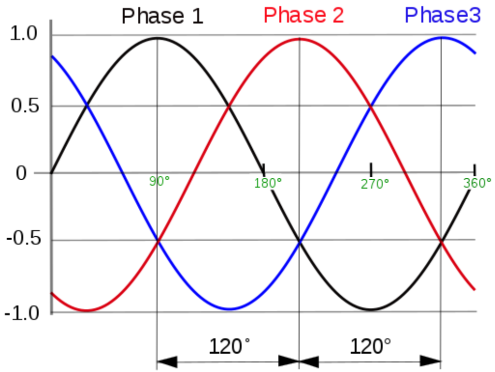
$$
\begin{aligned}
P_i + jQ_i &= \frac{1}{R + jX} V_i^2 \left[j(\theta_i - \theta_j) \right] \\
&\sim \frac{1}{jX} V_i^2 \left[j(\theta_i - \theta_j) \right] \\
&= \frac{V_i^2}{X} (\theta_i - \theta_j)
\end{aligned}
$$

Note that ignoring R means that we ignore resistive losses in the transmission lines and also since $Q_i \sim 0$, we ignore the flow of reactive power. Finally we absorb the voltage into the definition of the **per unit** reactance $x_\ell = \frac{X}{V_i^2}$ to get

$$
f_\ell = P_i = -P_j = \frac{\theta_i - \theta_j}{x_\ell}
$$

Three-phase power

Electricity is generally generated simultaneously in 3 separate circuits separate by 120 degrees or $\frac{2\pi}{3}$



In your plug, you only see one phase, but your oven may use all three phases.

Source: Wikipedia

Three-phase power

Why three phases? This was settled in the late 1880s.

1. The total power delivery is constant

$$\frac{d}{dt}P(t) = \frac{d}{dt}\left[P_a(t) + P_b(t) + P_c(t)\right] = 0$$

This reduces mechanical stress on generators and motors.

2. The sum of voltages and currents is zero, so no return path required! Saving on materials.

Both facts follow from

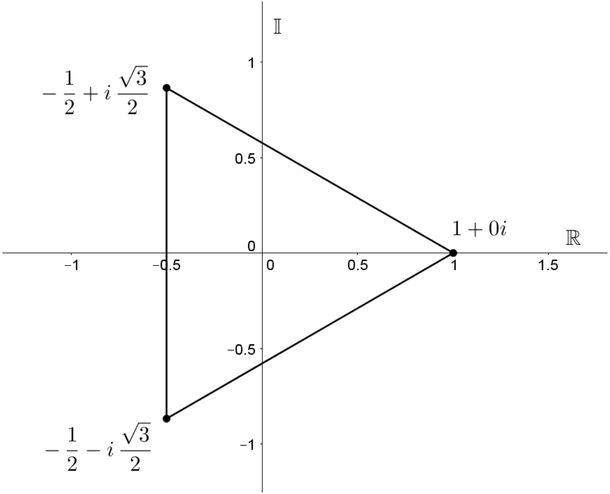
$$\sum_{k=0}^{N-1} e^{j\frac{2\pi k}{N}} = 0$$

for $N > 1$.

3. Why $N = 3$ rather than $N = 2$? Allows directional rotating fields for induction motors (thanks Tesla!).

Roots of unity for $N = 3$

For $N = 3$, check they add up to zero:



Source: **brilliant.org**

Rotating field in a three-phase induction motor

A brilliant insight (credited to Tesla, but the history is **complicated**) was that with three-phase power, you can place your wires spaced at $2\pi/3$ to create a **rotating** magnetic field

https://www.youtube.com/watch?v=LtJoJBUSe28

which can then induce a current in a rotor cage, which then experiences a torque thanks to the magnetic field: this is the principle of the **induction motor**.

It would not be possible to create such a rotating field with a single-phase or two-phase system.

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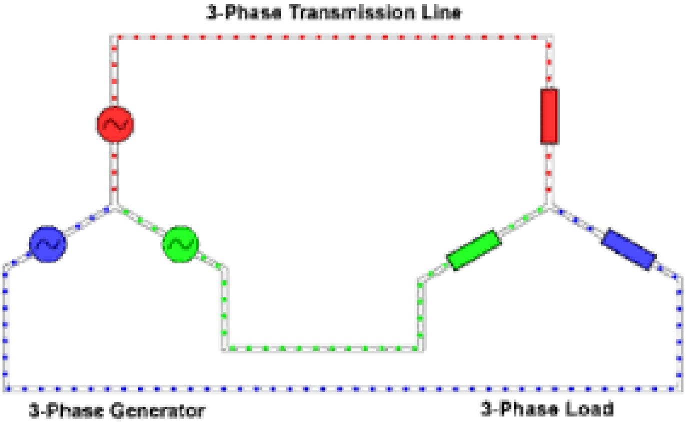

Source: Wikipedia