Energy System Modelling
Summer Semester 2018, Lecture 5

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Optimisation: Motivation

What to do about variable renewables?

Backup energy costs money and may also cause CO_2 emissions.

Curtailing renewable energy is also a waste.

We consider **four options** to deal with variable renewables:

1. Smoothing stochastic variations of renewable feed-in over **larger areas**, e.g. the whole of European continent.
2. Using **storage** to shift energy from times of surplus to deficit.
3. **Shifting demand** to different times, when renewables are abundant.
4. Consuming the electricity in **other sectors**, e.g. transport or heating.

Optimisation in energy networks is a tool to assess these options.

Why optimisation?

In the energy system we have lots of **degrees of freedom**:

1. Power plant and storage dispatch
2. Renewables curtailment
3. Dispatch of network elements (e.g. High Voltage Direct Current (HVDC) lines)
4. Capacities of everything when considering investment

but we also have to respect **physical constraints**:

1. Meet energy demand
2. Do not overload generators or storage
3. Do not overload network

and we want to do this while **minimising costs**. Solution: **optimisation**.

Optimisation: Introduction

A simple optimisation problem

Consider the following problem. We have a function $f(x, y)$ of two variables $x, y \in \mathbb{R}$

$$f(x, y) = 3x$$

and we want to find the maximum of this function in the $x - y$ plane

$$\max_{x,y\in\mathbb{R}} f(x, y)$$

subject to the following constraints

$$x + y \leq 4 \tag{1}$$

$$x \geq 0 \tag{2}$$

$$y \geq 1 \tag{3}$$

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$$x + y \leq 4 \quad (1)$$

$$x \geq 0 \quad (2)$$

$$y \geq 1 \quad (3)$$

Optimal solution: $x^* = 3, y^* = 1, f(x^*, y^*) = 9.$

Another simple optimisation problem

We can also have equality constraints. Consider the maximum of this function in the $x - y - z$ space

$$\max_{x,y,z \in \mathbb{R}} f(x, y, z) = (3x + 5z)$$

subject to the following constraints

$$\begin{aligned} x + y &\leq 4 \\ x &\geq 0 \\ y &\geq 1 \\ z &= 2 \end{aligned}$$

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Optimal solution: $x^* = 3, y^* = 1, z^* = 2, f(x^*, y^*, z^*) = 19.$

Energy system mapping to an optimisation problem

This optimisation problem has the same basic form as our energy system considerations:

Objective function to minimise	$\leftrightarrow$	**Minimise total costs**
Optimisation variables	$\leftrightarrow$	**Physical degrees of freedom (power plant dispatch, etc.)**
Constraints	$\leftrightarrow$	**Physical constraints (overloading, etc.)**

Before we apply optimisation to the energy system, we'll do some **theory**.

Optimisation: Theory

Optimisation problem

We have an **objective function** $f : \mathbb{R}^k \to \mathbb{R}$

$$\max_x f(x)$$

[$x = (x_1, \ldots x_k)$] subject to some **constraints** within $\mathbb{R}^k$:

$$\begin{array}{llll} g_i(x) = c_i & \leftrightarrow & \lambda_i & i = 1, \ldots n \\ h_j(x) \leq d_j & \leftrightarrow & \mu_j & j = 1, \ldots m \end{array}$$

λ_i and μ_j are the **KKT multipliers** (basically Lagrange multipliers) we introduce for each constraint equation; it measures the change in the objective value of the optimal solution obtained by relaxing the constraint (shadow price).

Feasibility

The space $X \subset \mathbb{R}^k$ which satisfies

$$\begin{aligned} g_i(x) &= c_i & \leftrightarrow \quad & \lambda_i & i &= 1, \dots n \\ h_j(x) &\leq d_j & \leftrightarrow \quad & \mu_j & j &= 1, \dots m \end{aligned}$$

is called the **feasible space**.

It will have dimension lower than k if there are independent equality constraints.

It may also be empty (e.g. $x \geq 1, x \leq 0$ in $\mathbb{R}$), in which case the optimisation problem is called **infeasible**.

It can be **convex** or **non-convex**.

If all the constraints are affine, then the feasible space is a convex polygon.

Lagrangian

We now study the **Lagrangian function**

$$\mathcal{L}(x, \lambda, \mu) = f(x) - \sum_i \lambda_i \left[g_i(x) - c_i\right] - \sum_j \mu_j \left[h_j(x) - d_j\right]$$

We've built this function using the variables λ_i and μ_j to better understand the optimal solution of $f(x)$ given the constraints.

The optima of $\mathcal{L}(x, \lambda, \mu)$ tell us important information about the optima of $f(x)$ given the constraints.

It is entirely analogous to the physics Lagrangian $L(x, \dot{x}, \lambda)$ except we have no explicit time dependence $\dot{x}$ and we have additional constraints which are inequalities.

We can already see that if $\frac{\partial \mathcal{L}}{\partial \lambda_i} = 0$ then the equality constraint $g_i(x) = c$ will be satisfied.

[Beware: $\pm$ signs appear differently in literature, but have been chosen here such that $\lambda_i = \frac{\partial \mathcal{L}}{\partial c_i}$ and $\mu_j = \frac{\partial \mathcal{L}}{\partial d_j}$.]

KKT conditions

The **Karush-Kuhn-Tucker (KKT) conditions** are necessary conditions that an optimal solution x^*, μ^*, λ^* always satisfies (up to some regularity conditions):

1. **Stationarity**: For $l = 1, \dots k$

$$\frac{\partial \mathcal{L}}{\partial x_l} = \frac{\partial f}{\partial x_l} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial x_l} - \sum_j \mu_j^* \frac{\partial h_j}{\partial x_l} = 0$$

2. **Primal feasibility**:

$$g_i(x^*) = c_i$$
$$h_j(x^*) \leq d_j$$

3. **Dual feasibility**: $\mu_j^* \geq 0$

4. **Complementary slackness**: $\mu_j^*(h_j(x^*) - d_j) = 0$

Complementarity slackness for inequality constraints

We have for each inequality constraint

$$\mu_j^* \geq 0$$
$$\mu_j^*(h_j(x^*) - d_j) = 0$$

So **either** the inequality constraint is binding

$$h_j(x^*) = d_j$$

and we have $\mu_j^* \geq 0$.

Or the inequality constraint is NOT binding

$$h_j(x^*) < d_j$$

and we therefore MUST have $\mu_j^* = 0$.

If the inequality constraint is non-binding, we can remove it from the optimisation problem, since it has no effect on the optimal solution.

Nota Bene

1. The KKT conditions are only **sufficient** for optimality of the solution under certain conditions, e.g. linearity of the problem.

2. Since at the optimal solution we have $g_i(x^*) = c_i$ for equality constraints and $\mu_j^*(h_j(x^*) - d_j) = 0$, we have

$$\mathcal{L}(x^*, \lambda^*, \mu^*) = f(x^*)$$

Return to simple optimisation problem

We want to find the maximum of this function in the $x - y$ plane

$$\max_{x,y \in \mathbb{R}} f(x, y) = 3x$$

subject to the following constraints (now with KKT multipliers)

$$\begin{aligned} x + y \leq 4 \quad &\leftrightarrow \quad \mu_1 \\ -x \leq 0 \quad &\leftrightarrow \quad \mu_2 \\ -y \leq -1 \quad &\leftrightarrow \quad \mu_3 \end{aligned}$$

We know the optimal solution in the **primal variables** $x^* = 3, y^* = 1, f(x^*, y^*) = 9$.

What about the **dual variables** μ_i?

Since the second constraint is not binding, by complementarity $\mu_2^*(-x^* - 0) = 0$ we have $\mu_2^* = 0$. To find μ_1^* and μ_3^* we have to do more work.

Simple problem with KKT conditions

We use stationarity for the optimal point:

$$0 = \frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial x} - \sum_j \mu_j^* \frac{\partial h_j}{\partial x} = 3 - \mu_1 + \mu_2$$

$$0 = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial f}{\partial y} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial y} - \sum_j \mu_j^* \frac{\partial h_j}{\partial y} = -\mu_1 + \mu_3$$

From which we find:

$$\mu_1^* = 3 - \mu_2^* = 3$$

$$\mu_3^* = \mu_1^* = 3$$

Check interpretation: $\mu_j = \frac{\partial \mathcal{L}}{\partial d_j}$ with $d_j \to d_j + \varepsilon$.

Return to another simple optimisation problem

We want to find the maximum of this function in the $x - y - z$ space

$$\max_{x,y,z\in\mathbb{R}} f(x,y) = 3x + 5z$$

subject to the following constraints (now with KKT multipliers)

$$\begin{aligned} x + y \leq 4 \quad &\leftrightarrow \quad \mu_1 \\ -x \leq 0 \quad &\leftrightarrow \quad \mu_2 \\ -y \leq -1 \quad &\leftrightarrow \quad \mu_3 \\ z = 2 \quad &\leftrightarrow \quad \lambda \end{aligned}$$

We know the optimal solution in the **primal variables** $x^* = 3, y^* = 1, z^* = 2, f(x^*, y^*, z^*) = 19$.

What about the **dual variables** μ_i, λ?

We get same solutions to $\mu_1^* = 3, \mu_2^* = 0, \mu_3^* = 3$ because they're not coupled to z direction. What about λ^*?

Another simple problem with KKT conditions

We use stationarity for the optimal point:

$$0 = \frac{\partial \mathcal{L}}{\partial z} = \frac{\partial f}{\partial z} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial z} - \sum_j \mu_j^* \frac{\partial h_j}{\partial z} = 5 - \lambda^*$$

From which we find:

$$\lambda^* = 5$$

Check interpretation: $\lambda_i = \frac{\partial \mathcal{L}}{\partial c_i}$ with $c_i \to c_i + \varepsilon$.