Energy System Modelling
Summer Semester 2019, Lecture 2

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Karlsruhe Institute of Technology

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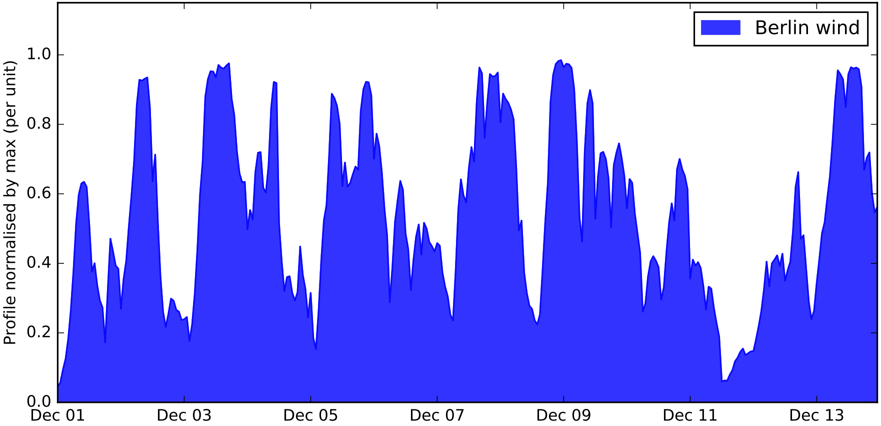
Single location versus country versus continent

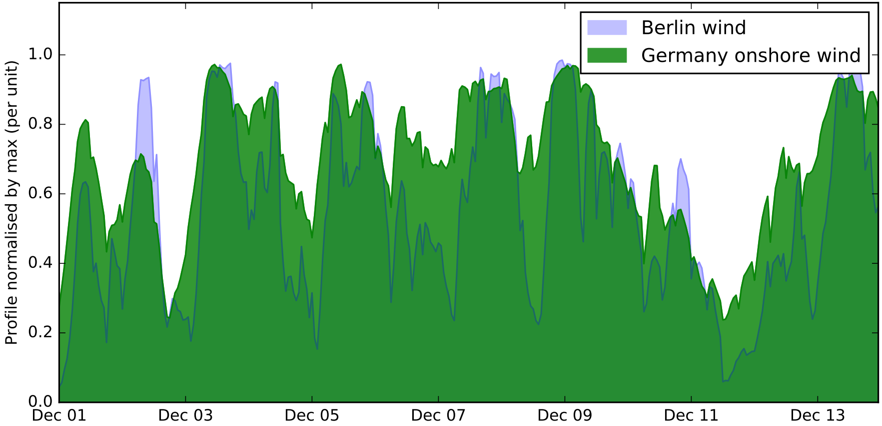
Variability: Single wind site in Berlin

Looking at the wind output of a single wind plant over two weeks, it is highly variable, frequently dropping close to zero and fluctuating strongly.



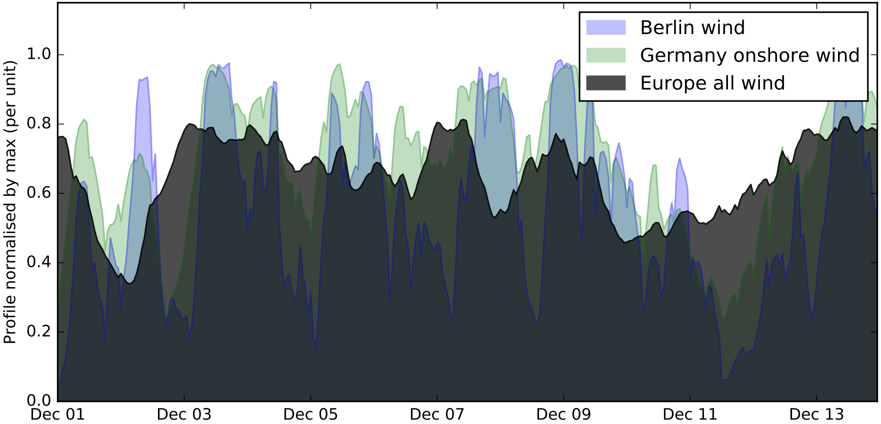
Variability: Single country: Germany

For a whole country like Germany this results in valleys and peaks that are somewhat smoother, but the profile still frequently drops close to zero.



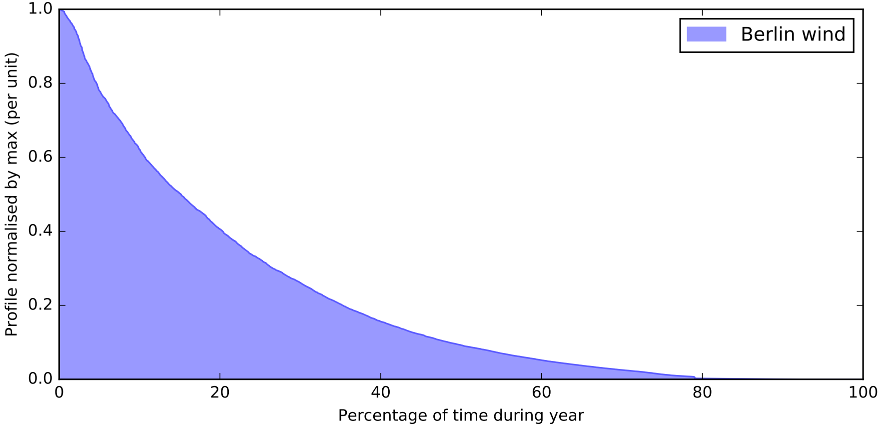
Variability: A continent: Europe

If we can integrate the feed-in of wind turbines across the European continent, the feed-in is considerably smoother: we've eliminated most valleys and peaks.



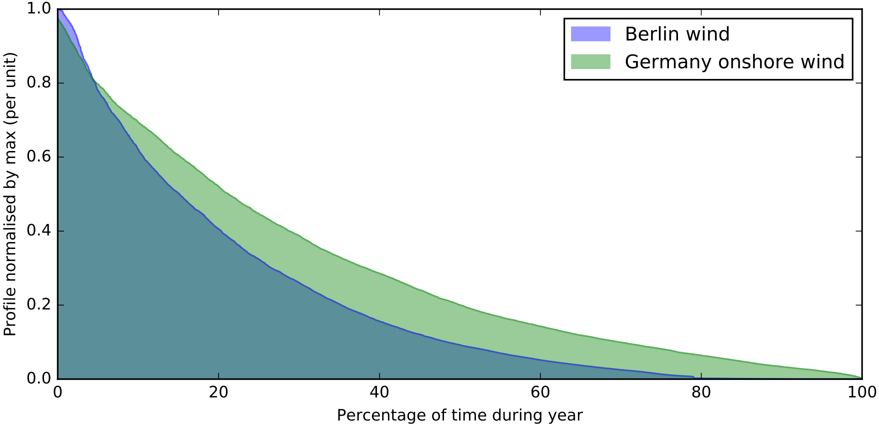
Duration curve: Berlin

A **duration curve** shows the feed-in for the whole year, re-ordered by from highest to lowest value. For a single location there are many hours with no feed-in.



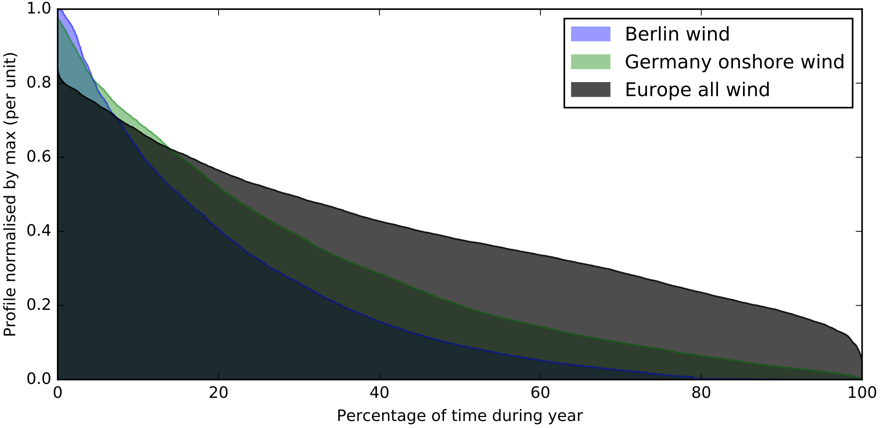
Duration curve: Germany

For a whole country there are fewer peaks and fewer hours with no feed-in.



Duration curve: Europe

For the whole of Europe there are no times with zero feed-in.



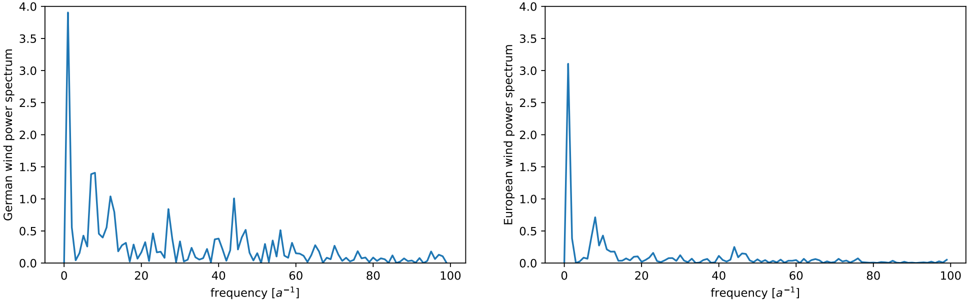
Statistical comparison

Area	Mean	Standard deviation
Berlin	0.21	0.26
Germany	0.26	0.24
Europe (including offshore)	0.36	0.19

Conclusion: Wind generation has much lower variability if you integrate it over a continent-sized area.

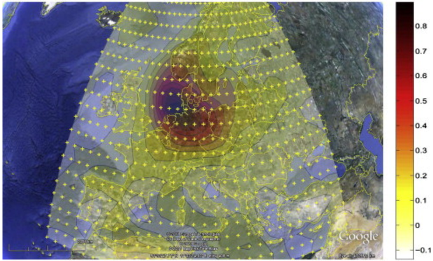
Statistical comparison

The **synoptic** (2-3 weeks) variations in the Fourier spectrum are also suppressed between Germany (left) and the Europe profile (right), however the seasonal variations remain.

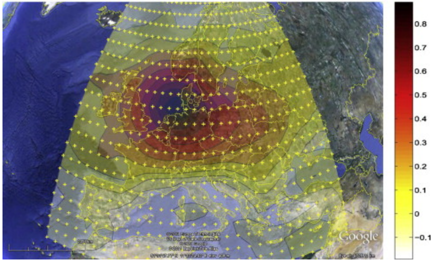


Why does this work? Consider the correlation length of wind

(a) Summer-day



(b) Winter-day



- The Pearson correlation coefficient of wind time series with a point in northern Germany decays with distance.
- Determine the **correlation length** L by fitting the function:

$$\rho \sim e^{-\frac{x}{L}}$$

  to the radial decay with distance x.
- Typically correlation lengths for wind are around 400 – 600 km. Smoothing requires aggregating uncorrelated sources, so need a bigger area, i.e. a continent (Europe is about 3500 km tall and 3100 km wide).

Mismatch between load and renewables

How does the mismatch change as we integrate over larger areas?

If we have for each time t a demand of d_t and a 'per unit' availability w_t for wind and s_t for solar, then if we have W MW of wind and S MW of solar, the effective **residual load** or **mismatch** is

$$m_t = d_t - Ww_t - Ss_t$$

We choose W and S such that on **average** we cover all the load

$$\langle m_t \rangle = 0$$

and so that the 70% of the energy comes from wind and 30% from solar ($W = 147$ GW and $S = 135$ GW for Germany).

This means

$$W\langle w_t \rangle = 0.7\langle d_t \rangle \qquad S\langle s_t \rangle = 0.3\langle d_t \rangle$$

Mismatch between load and renewables

Let p_t be the balance of power at each time. Because we cannot create or destroy energy, we need $p_t = 0$ at all times.

If the mismatch is positive $m_t > 0$, then we need **backup power** $b_t = m_t$ to cover the load in the absence of renewables, so that

$$p_t = b_t - m_t = b_t - d_t + Ww_t + Ss_t = 0$$

If the mismatch is negative $m_t < 0$ then we need **curtailment** $c_t = -m_t$ to reduce the excess feed-in from renewables, so that

$$p_t = -m_t - c_t = -c_t - d_t + Ww_t + Ss_t = 0$$

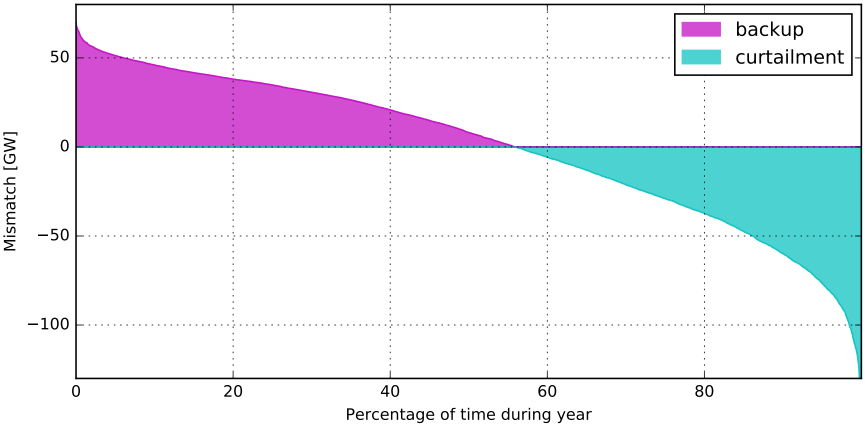
At any one time we have either backup or curtailment

$$p_t = b_t - m_t - c_t = Ww_t + Ss_t + b_t - d_t - c_t = 0$$

Mismatch for Germany

Backup generation needed for 31% of the total load.

Peak mismatch is 91% of peak load (around 80 GW).

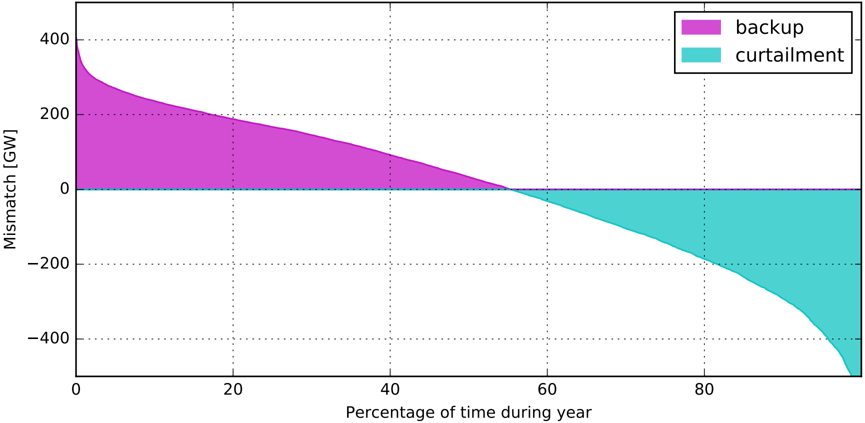


Mismatch for Europe

Requires 750 GW each of onshore wind and solar.

Backup generation needed for only 24% of the total load.

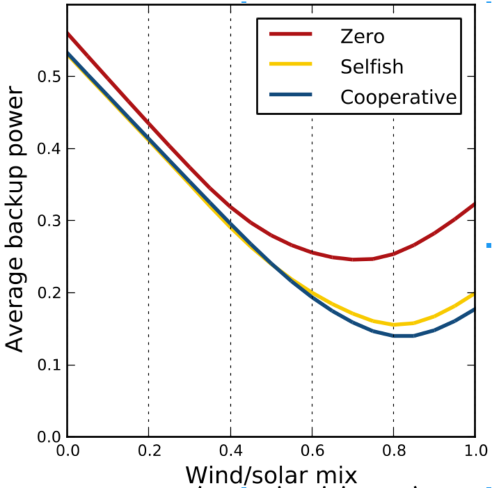
Peak mismatch is 79% of peak load (around 500 GW).

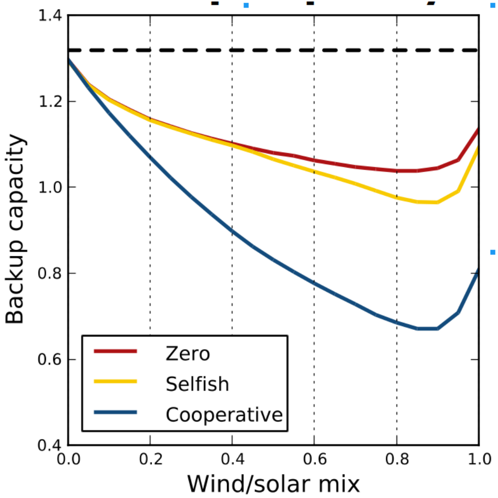


Conclusions

- Integration over a larger area smooths out the fluctuations of renewables, particularly wind
- Wind backs up wind
- This means we need **less backup energy**.
- and **less backup capacity**.

'Cost-optimal design of a simplified, highly renewable pan-European electricity system' by Rolando A. Rodriguez, Sarah Becker, Martin Greiner, Energy 83 (2015) 658-668

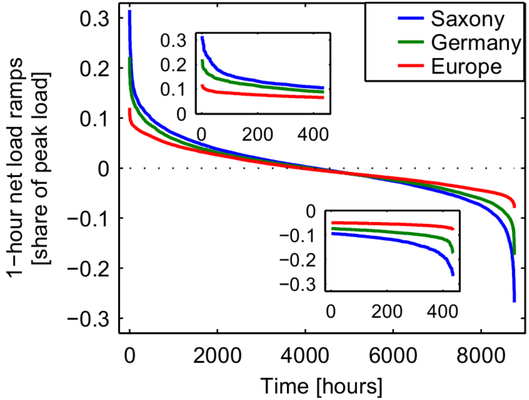




Flexibility Requirements

'Integration of wind and solar power in Europe: Assessment of flexibility requirements' by Huber, Dimkova, Hamacher, Energy 69 (2014) 236e246

1-hour net load ramp duration curves at the regional, country and European spatial scales at 50% share of renewables and 20% PV in the wind/PV mix for the meteorological year 2009.



Big Caveat

There is a big caveat to this analysis.

We've assumed that we can move power around Europe without penalty.

However, in reality, we can only transport within restrictions of the power network.

In general we will have different power imbalances $p_{i,t}$ at each location/node i and instead of $p_t = 0$ we will have

$$\sum_i p_{i,t} = 0$$

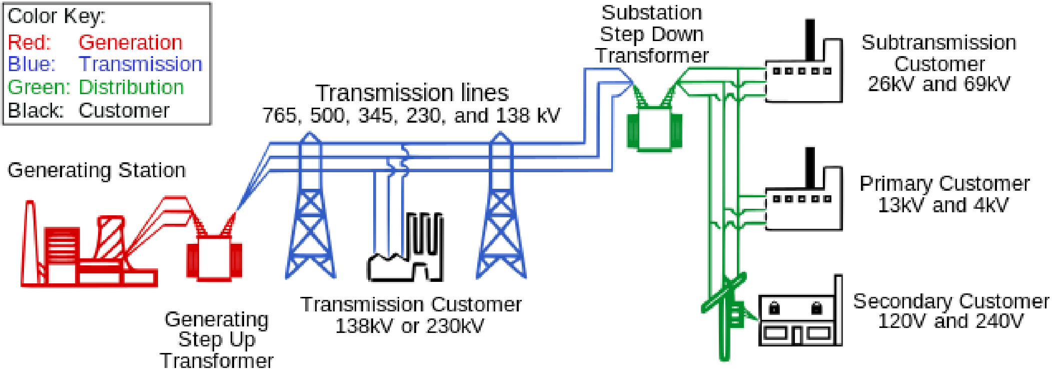
(neglecting power losses in the network).

Moving excess power to locations of consumption is the role of the network.

Networks

Electricity Transport from Generators to Consumers

Electricity can be transported over long distances with low losses using the high voltage transmission grid:



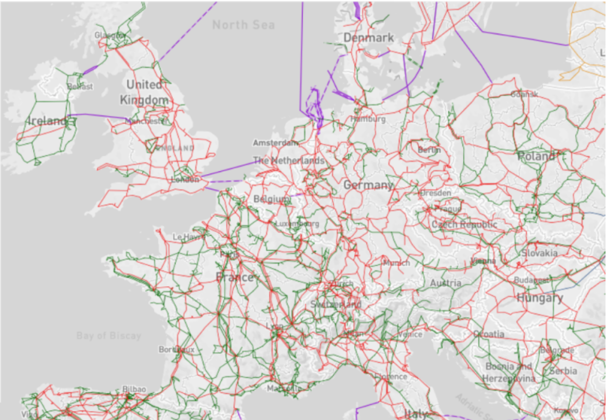
Usually in houses the voltage is 230 V, but in the transmission grid it is transformed up to hundreds of thousands of Volts.

Source: Wikipedia

European transmission network

Flows in the European transmission network must respect both Kirchoff's laws for physical flow and the thermal and/or other limits of the power lines.

Taking account of network flows and constraints in the electricity market is a major and exciting topic at the moment.

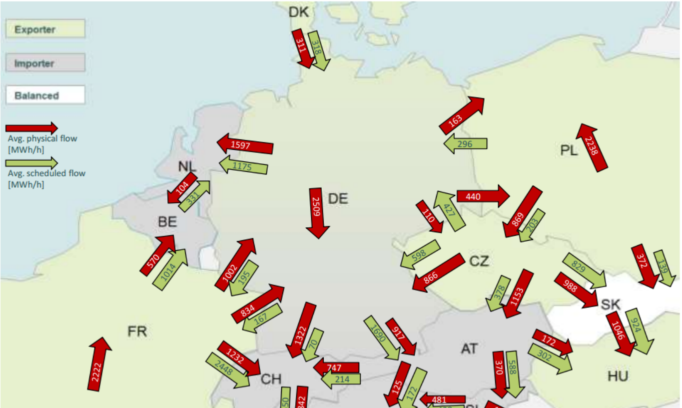


Source: ENTSO-E

Network Bottlenecks and Loop Flows

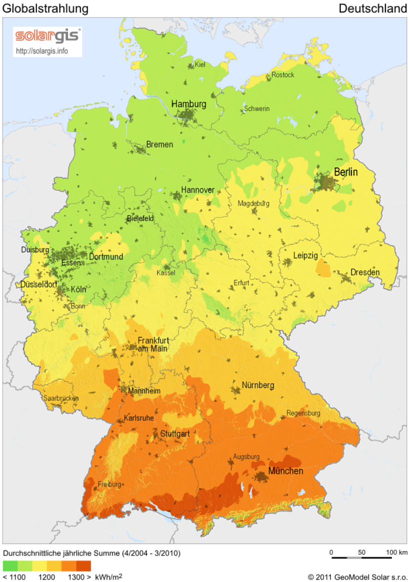
Electricity is traded in large market zones. Power trades between zones (“scheduled flows”) do not always correspond to what flows according to the network physics (“physical flows”). This leads to political tension as wind from Northern Germany flows to Southern Germany via Poland and the Czech Republic.

Figure 7: Average physical and scheduled flows [MWh/h], 01.01.2011 – 31.12.2012



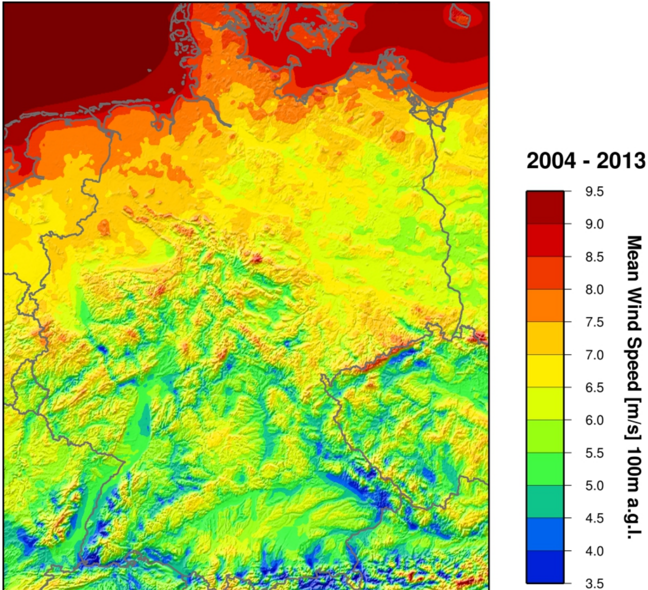
Source: THEMA Consulting Group

Solar resource distribution in Germany



- Solar insolation at top of atmosphere is on average 1361 W/m^2 (orbit is elliptical).
- In Germany average insolation on a horizonal surface is around 1200 kWh/m^2.
- A 1 kW solar panel (around 7 m^2) will generate around 1000 kWh/a.

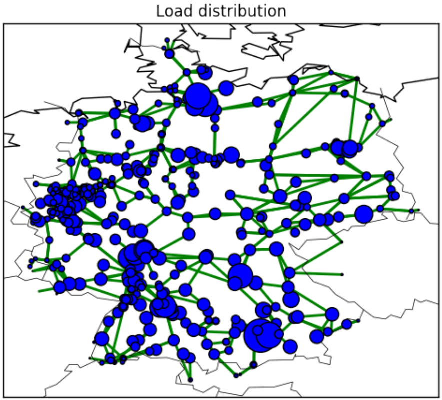
Wind resource distribution in Germany

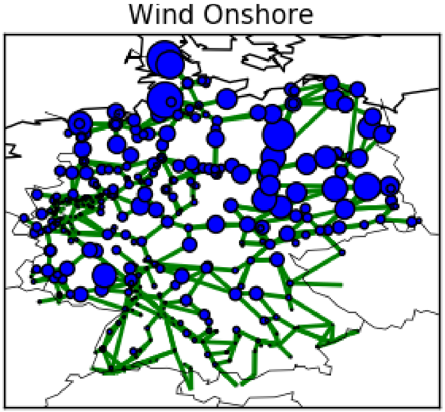


- Best wind speeds in Germany in North and on hills.
- In theory power output goes like cube $\propto v^3$ of wind speed v.
- In practice power-speed relationship is only partially cubic.

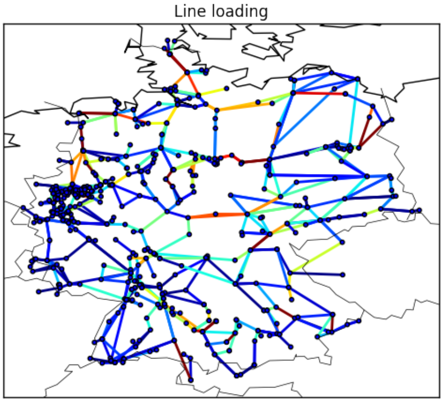
The Problem

Renewables are not always located near demand centres, as in this example from Germany.





The Problem

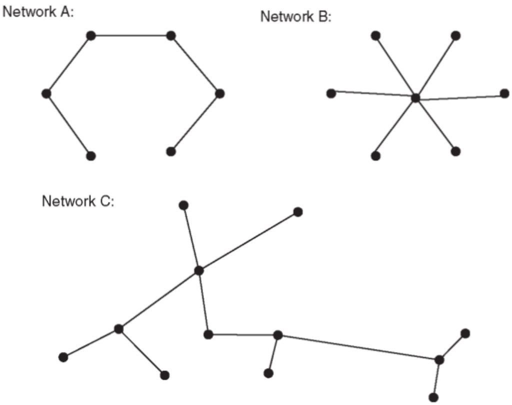


- This leads to **overloaded lines** in the middle of Germany, which cannot transport all the wind energy from North Germany to the load in South Germany
- It also overloads lines in neighbouring countries due to **loop flows** (unplanned physical flows 'according to least resistance' which do not correspond to traded flows)
- It also **blocks imports and exports** with neighbouring countries, e.g. Denmark

Different types of networks: radial networks

In a **radial** or **tree-like** network there is only one path between any two nodes on the network.

The power flow is thus completely determined by the nodal power imbalances.

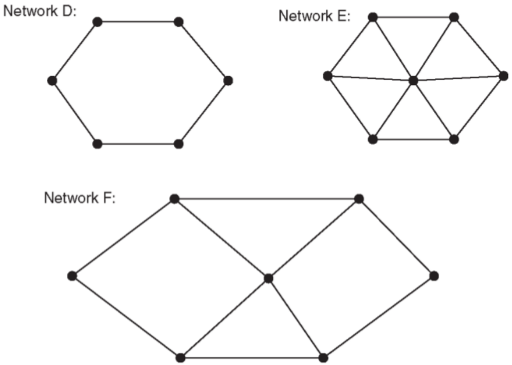


Source: Biggar & Hesamzadeh

Different types of networks: meshed networks

In a **meshed** network there are at least two nodes with multiple paths between them.

The power flow is now not completely determined. We need new information: the impedances in the network.



Source: Biggar & Hesamzadeh

Graph Theory

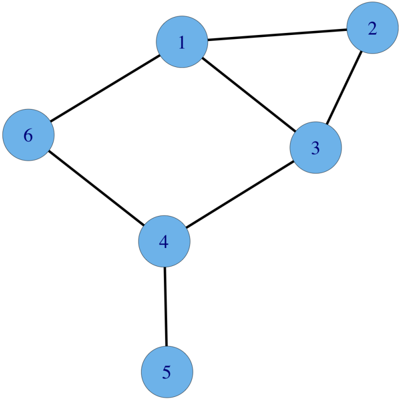
Definition of a network

Our definition (Newman): A **network** (graph) is a collection of **vertices** (nodes) joined by **edges** (links).

More precise definition (Bollobàs): A graph G is an ordered pair of disjoint sets (V, E) such that E (the edges) is a subset of the set $V^{(2)}$ of unordered pairs of V (the vertices).

Edge list representation

- Vertices:
  1,2,3,4,5,6
- Edges:
  (1,2), (1,3), (1,6), (2,3), (3,4), (4,5), (4,6)



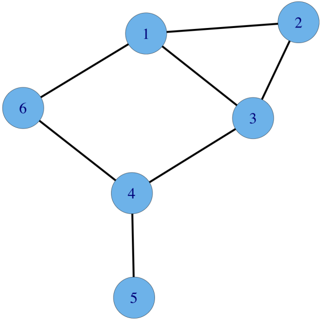
Definition from graph theory:

- $n = 6$ vertices: **order** of the graph
- $m = 7$ edges: **size** of the graph

Adjacency matrix A

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge between vertices i and j} \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

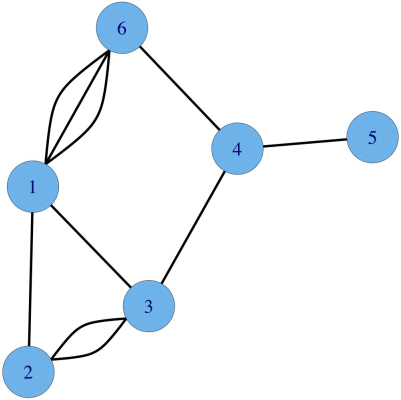


- Diagonal elements are zero.
- Symmetric matrix.
- If there are N vertices, it's an $N \times N$ matrix.

Multigraph

There can be more than one edge between a pair of vertices.

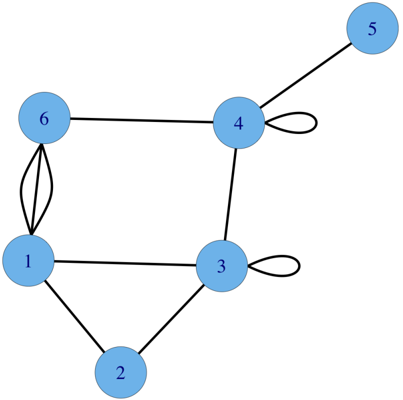
$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 3 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



Self-edges

There can be **self-edges** (also called self-loops).

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 3 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

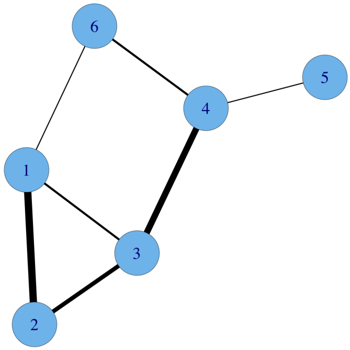


- Diagonal elements can be non-zero:
  Definition: $A_{ii} = 2$ for one self-edge.

Weighted networks

We can assign a **weight** or **strength** assigned to each edge.

$$\mathbf{A} = \begin{pmatrix} 0 & 1.4 & 0.4 & 0 & 0 & 0.8 \\ 1.4 & 0 & 1.2 & 0 & 0 & 0 \\ 0.4 & 1.2 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0.8 & 0 & 0 & 0.4 & 0 & 0 \end{pmatrix}$$



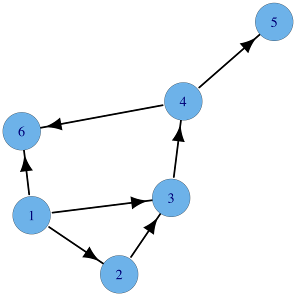
Weights can be both positive or negative.

Directed Networks (Digraphs)

A graph is **directed** if each edge is pointing from one vertex to another (**directed edge**).

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge from } j \text{ to } i \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



In general the adjacency matrix of a directed network is asymmetric.

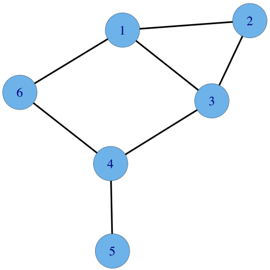
Degree

- The **degree** k_i of a vertex i is defined as the number of edges connected to i.
- Average degree of the network: $\langle k \rangle$.

In terms of the adjacency matrix $\mathbf{A}$:

$$k_i = \sum_{j=1}^{n} A_{ij} \quad , \quad \langle k \rangle = \frac{1}{n} \sum_{i} k_i = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \ .$$

$$k_5 = 1$$
$$k_2 = k_6 = 2$$
$$k_1 = k_3 = k_4 = 3$$
$$\langle k \rangle = 2.33$$



Examples

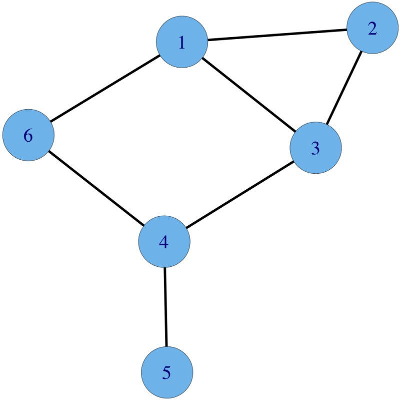
NETWORK	NODES	LINKS	DIRECTED UNDIRECTED	N	L	⟨k⟩
Internet	Routers	Internet connections	Undirected	192,244	609,066	6.34
WWW	Webpages	Links	Directed	325,729	1,497,134	4.60
Power Grid	Power plants, transformers	Cables	Undirected	4,941	6,594	2.67
Mobile Phone Calls	Subscribers	Calls	Directed	36,595	91,826	2.51
Email	Email addresses	Emails	Directed	57,194	103,731	1.81
Science Collaboration	Scientists	Co-authorship	Undirected	23,133	93,439	8.08
Actor Network	Actors	Co-acting	Undirected	702,388	29,397,908	83.71
Citation Network	Paper	Citations	Directed	449,673	4,689,479	10.43
E. Coli Metabolism	Metabolites	Chemical reactions	Directed	1,039	5,802	5.58
Protein Interactions	Proteins	Binding interactions	Undirected	2,018	2,930	2.90

(from the free textbook "Network Science")

Degree matrix D

$$D_{ij} = \begin{cases} k_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

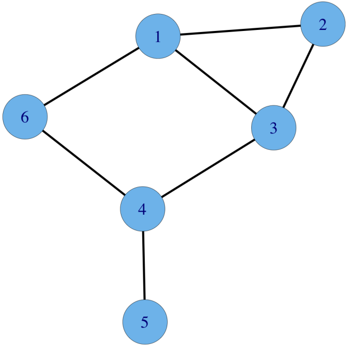


Laplacian L

The **Laplacian matrix** is defined for an undirected graph by

$$\mathbf{L} = \mathbf{D} - \mathbf{A}$$

$$\mathbf{L} = \begin{pmatrix} 3 & -1 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$



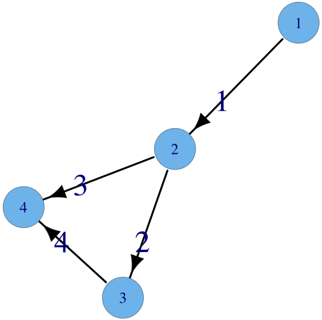
- $\mathbf{L}$ inherits symmetry from $\mathbf{D}$ and $\mathbf{A}$.
- The number of zero eigenvalues equals the number of connected components.
- For a set of connected nodes I, $\sum_{i \in I} L_{ij} = 0 \ \forall j$.

The incidence matrix

For a directed graph (every edge has an orientation) $G = (V, E)$ with N nodes and L edges, the node-edge **incidence matrix** $K \in \mathbb{R}^{N \times L}$ has components

$$K_{i\ell} = \begin{cases} 1 & \text{if edge } \ell \text{ starts at node } i \\ -1 & \text{if edge } \ell \text{ ends at node } i \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{K} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$



Incidence matrix properties

The incidence matrix has several important properties.

First, for a given edge ℓ, the corresponding column sums to zero $\sum_i K_{i\ell} = 0$, since every edge starts at some node ($+1$) and ends at some node (-1).

The row corresponding to each node i tells you which edges start there ($+1$) and which edges end there (-1).

It is related to the Laplacian matrix by

$$L = KK^t$$

Check the definitions agree:

$$L_{ij} = \sum_\ell K_{i\ell} K_{j\ell}$$

for $i = j$ and $i \neq j$.

The kernel of the incidence matrix

The kernel of $K_{i\ell}$, i.e. particular combinations of edges which are annihilated by K, has a very special meaning.

Consider the combination of edges $(0, 1, -1, 1)^t$

$$\mathbf{K} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

This corresponds to a **closed cycle** in the graph, since the edges form a path that returns to its starting point. Each point in the cycle has an edge that ends there and an edge that starts there.

The matrix K can be interpreted as a **boundary operator**. A cycle has no boundary in 0-d. There is a general theory called **homology theory**, which can compute topological invariants of manifolds called **homology groups**.

Cycle matrix

We can organise the cycles in a matrix $C_{\ell c}$, where c labels each cycle.

We have

$$KC = 0$$

by definition of C being in the kernel.

The image of K has dimension $N - 1$ (i.e. the rank of K) for a connected graph, since the space spanned by the columns of K can only reach differences between nodes and never then N-length vector $(1, 1, \ldots 1)^t$.

By the rank-nullity theorem for K we have

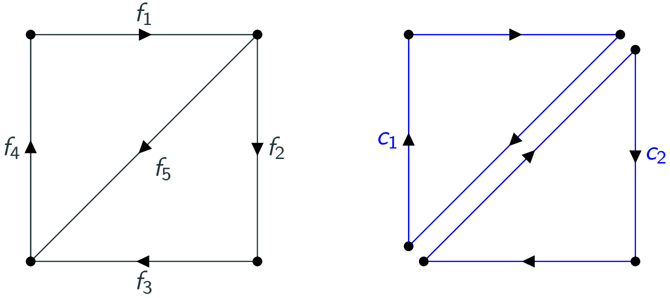
$$L = \dim \mathrm{im} K + \dim \ker K$$

so the number of cycles, i.e. the dimension of the kernel (nullity) of K is $L - N + 1$. If the connected graph has no cycles, i.e. it is a tree, then $L = N - 1$.

In our case $L = 4$, $N = 4$ so there is only 1 cycle

$$\mathbf{C} = (0, 1, -1, 1)^t$$

Independent basis of cycles



Two independent cycles:

$c_1 = f_1 + f_5 + f_4$

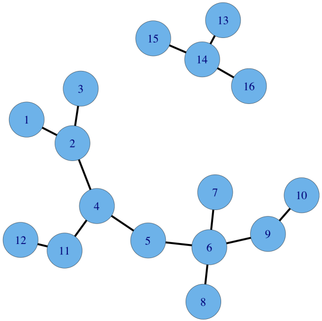
$c_2 = f_2 + f_3 + -f_5$

The outer cycle is not independent:

$c_3 = f_1 + f_2 + f_3 + f_4 = c_1 + c_2$

Trees

- A collection of trees is called a **forest**.
- Trees play an import role for random graph models.
- In a tree, there is exactly one path between any pair of vertices.
- A tree of N vertices always has exactly $N - 1$ edges.
- Any connected network with N vertices and $N - 1$ edges is a tree.
- Trees have **no cycles**.

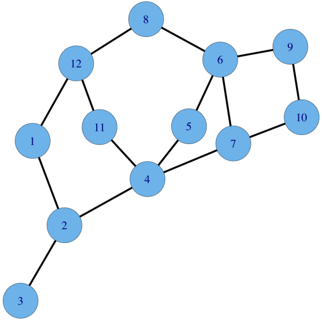


Planar networks

A **planar network** is a network that can be drawn on a plane without having any edges cross.

Examples:

- Trees
- Road networks (approximately)
- Power grids (approximately)
- Shared borders between countries, etc.



Paths

- Route through the network, from vertex to vertex along the edges
- Defined for both directed and undirected networks
- Special case: self-avoiding paths
- **Length** of a path: number of edges along the path ("hops")
- Number of paths of length r between vertices i and j:

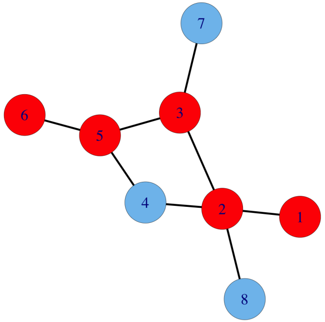
$$N_{ij}^{(r)} = [\mathbf{A}^r]_{ij}$$

- Total number L_r of loops of length r anywhere in the network:

$$L_r = \sum_{i=1}^{n} [\mathbf{A}^r]_{ii} = \mathrm{Tr}\mathbf{A}^r \ .$$

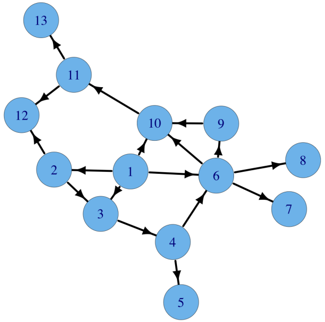
Geodesic / shortest paths

- A path between two vertices such that no shorter path exists
- Geodesic distance between vertices i and j is the smallest value of r such that $[\mathbf{A}^r]_{ij} > 0$.
- Self-avoiding
- In general not unique
- **Diameter** of a network: Length of the longest geodesic path between any pair of vertices



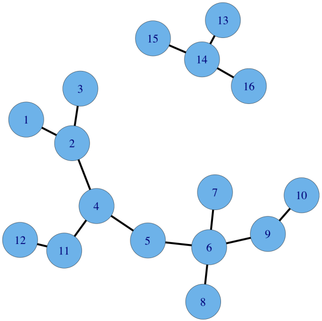
Acyclic directed network

- Directed network without closed loops of edges (DAG)
- Examples: power flow in an electricity grid, citation network of papers
- Topological ordering: For every directed edge $i \to j$, vertex i comes before j in the ordering: (1,2,3,4,6,9,10,11,12,8,7,5,13)
- With a topological ordering, the adjacency matrix of an acyclic directed network is **strictly triangular**



Components of networks

- Subgroups of vertices with no connections between the respective groups
- **Disconnected** network
- Subgroups: **components**
- Adjacency matrix: Block-diagonal form



Why is it called the Laplacian?

What does this matrix have to do with the second-order Laplacian:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1}$$

from continuous physics?

On a 1d lattice, for each link (difference) from K^t get $u_i - u_{i-1} \sim \frac{d}{dx}$. From $L = KK^t$ get $2u_i - u_{i-1} - u_{i+1} \sim \frac{d^2}{dx^2}$.

Similarly for 2d lattice, from the Laplacian you get

$$4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} \sim \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2}$$

which is a second-order difference in both x and y directions.

In fact you can do interesting discrete physics with these matrices (more later...).

(Co)homology analogy

$$K \leftrightarrow \delta \text{ (1d boundary operator)}$$
$$K^t \leftrightarrow d \text{ (0d differential)}$$
$$L = KK^t \leftrightarrow \Delta = d * d \text{ (0d Laplacian)}$$

On a 1d lattice, for each link (difference) from K^t get $u_i - u_{i-1} \sim \frac{d}{dx}$. From $L = KK^t$ get $2u_i - u_{i-1} - u_{i+1} \sim \frac{d^2}{dx^2}$.

Similarly for 2d lattice.

Computing the Linear Power Flow

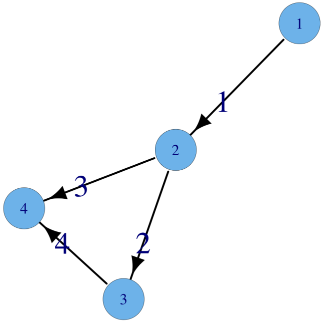
The goal of power flow analysis

The goal of a power/load flow analysis is to find the flows in the lines of a network given a power injection pattern at the nodes.
I.e. given power injection at the nodes

$$\mathbf{P}_i = \begin{pmatrix} 50 \\ 50 \\ 0 \\ -100 \end{pmatrix}$$

what are the flows in lines 1-4?

To find the flows, it is sufficient to know the **impedances** of the lines and the **voltages** at each node.



Framing the load flow problem

Suppose we have N nodes labelled by i, and L edges labelled by ℓ forming a directed graph G.

Suppose at each node we have a **power imbalance** p_i ($p_i > 0$ means its generating more than it consumes and $p_i < 0$ means it is consuming more than it).

Since we cannot create or destroy energy (and we're ignoring losses):

$$\sum_i p_i = 0$$

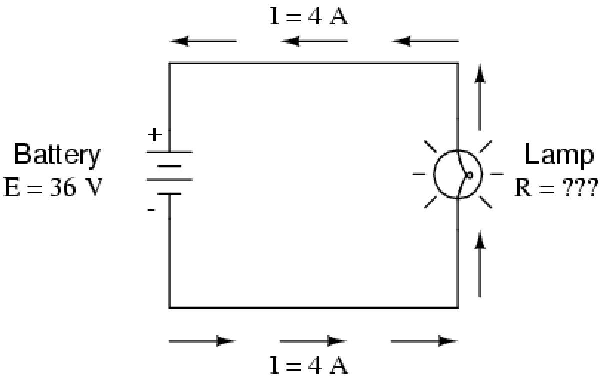
Question: How do the flows f_ℓ in the network relate to the nodal power imbalances?

Answer: According to the impedances (generalisation of resistance for oscillating voltage/current) and the corresponding voltages.

Ohm's Law

Ohm's Law: The potential difference (voltage) $V_1 - V_2$ across an ideal conductor is proportional to the current through it I. The constant of proportionality is called the **resistance**, R. Ohm's Law is thus:

$$V_1 - V_2 = I\,R$$



Analogy DC circuits to linear power flow

The equations for DC circuits and linear power flow in AC circuits are analogous:

$$I = \frac{V_i - V_j}{R} \qquad \leftrightarrow \qquad f_\ell = \frac{\theta_i - \theta_j}{x_\ell}$$

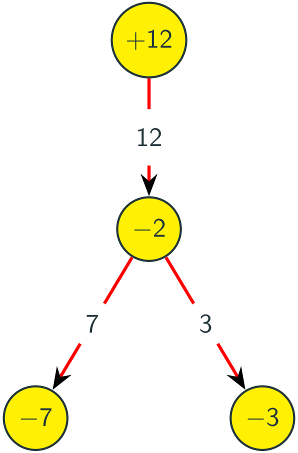
if we make the following identification:

Current flow I	$\leftrightarrow$	Active power flow f_ℓ
Potential/voltage V_i	$\leftrightarrow$	Voltage angle θ_i
Resistance R	$\leftrightarrow$	Reactance X

The simplifications that lead to the linear power flow will be explained in the next lecture.

Kirchhoff's Current Law (KCL)

KCL inforces energy conservation at each vertex (the power imbalance equals what goes out minus what comes in).



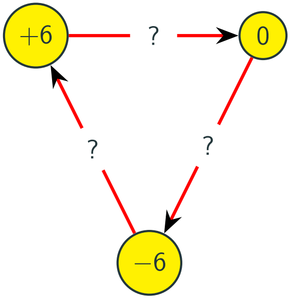
Kirchhoff's Current Law (KCL)

KCL says (in this linear setting) that the nodal power imbalance at node i is equal to the sum of direct flows arriving at the node. This can be expressed compactly with the incidence matrix

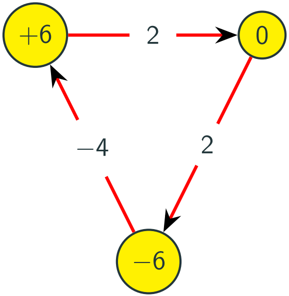
$$p_i = \sum_{\ell} K_{i\ell} f_\ell \qquad \forall i$$

Kirchhoff's Voltage Law (KVL)

KCL isn't enough to determine the flow as soon as there are **closed cycles** in the network. For this we need Ohm's law in combination with KVL: voltage differences around each cycle add up to zero.



For equal reactances for each edge:



NB: For directed graph, sign determines direction of flow.

Kirchhoff's Voltage Law (KVL)

KVL says that the sum of voltage differences across edges for any closed cycle must add up to zero.

If the voltage at any node is given by θ_i (this is infact the voltage **angle** - more next time) then the voltage difference across edge ℓ is

$$\sum_i K_{i\ell}\theta_i$$

And Kirchhoff's law can be expressed using the cycle matrix encoding of independent cycles

$$\sum_\ell C_{\ell c} \sum_i K_{i\ell}\theta_i = 0 \qquad \forall c$$

[Automatic, since we already said KC = 0.]

Kirchhoff's Voltage Law (KVL)

If we express the flow on each line in terms of the voltage angle (a relative of $V = IR$) then for a line ℓ with reactance x_ℓ

$$f_\ell = \frac{\theta_i - \theta_j}{x_\ell} = \frac{1}{x_\ell} \sum_i K_{i\ell} \theta_i$$

KVL now becomes

$$\sum_\ell C_{\ell c} x_\ell f_\ell = 0 \qquad \forall c$$

Solving the equations

If we combine

$$f_\ell = \frac{1}{x_\ell} \sum_i K_{i\ell} \theta_i$$

with Kirchhoff's Current Law we get

$$p_i = \sum_\ell K_{i\ell} f_\ell = \sum_\ell K_{i\ell} \frac{1}{x_\ell} \sum_j K_{j\ell} \theta_j$$

This is a **weighted Laplacian**. If we write $B_{k\ell}$ for the diagonal matrix with $B_{\ell\ell} = \frac{1}{x_\ell}$ then

$$L = KBK^t$$

and we get a **discrete Poisson equation** for the θ_i sourced by the p_i

$$p_i = \sum_j L_{ij} \theta_j$$

We can solve this for the θ_i and thus find the flows.