Energy System Modelling
Summer Semester 2019, Lecture 3

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Full power flow equations

Goal

Last time we said we can (in the linear approximation) express the flow on each line in terms of the voltage angle at the end buses (a relative of $V = IR$) for a line ℓ with reactance x_ℓ as

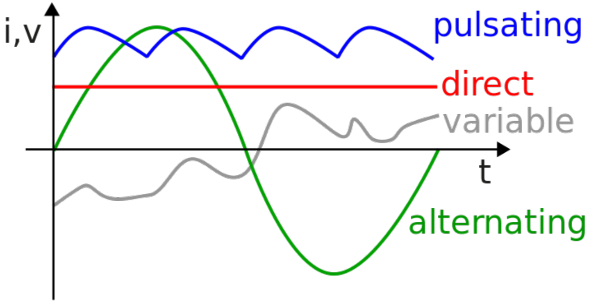
$$f_\ell = \frac{\theta_i - \theta_j}{x_\ell} = \frac{1}{x_\ell} \sum_i K_{i\ell} \theta_i$$

Now we explain where this comes from, and the linear approximation that leads to it.

This is also useful when we consider the synchronisation of oscillators later.

Alternating Current

The majority of electrical power, including what you get out of a wall plug, is transmitted as **Alternating Current (AC)**, i.e. both the voltage and current are sinusoidal waves.



Source: Wikipedia

[Some power is transmitted as **Direct Current (DC)** under bodies of water and indeed many electronic devices require DC (must convert AC to DC).]

Why alternating current?

Battle of currents! Edison versus Westinghouse/Tesla, etc.

https://en.wikipedia.org/wiki/War_of_Currents

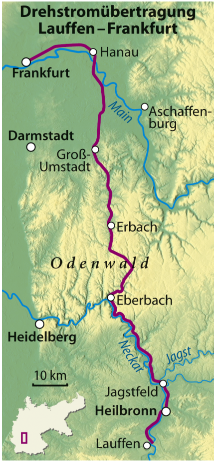
AC won, because it's easy to transform AC to a higher voltage, so you can transmit a given power with a lower current and thus avoid the I^2R resistive losses in power lines.

Reason: $\frac{d}{dt}$ in $\mathcal{E} = \frac{d\Phi}{dt}$; use a solenoid to induce a **fluctuating** magnetic field in another solenoid with a different number of turns, giving different potential difference.

Frequency of 50 Hz is uniform across Europe (except for train-electricity, e.g. in Germany 16.7 Hz). 60 Hz in USA, half of Japan, etc.

Frankfurt: Home of Long-Distance AC Transmission

First long-distance high-voltage alternating-current transmission in 1891 from hydro plant in Lauffen to Frankfurt for the Elektrotechnische Ausstellung (176 km, 15 kV).



Source: Wikipedia

Sinuisoidal waves

The voltage is usually written in terms of the frequency $\omega = 2\pi f$ and the **Root-Mean-Squared (RMS)** voltage magnitude V_{rms}

$$V(t) = V_{\mathrm{peak}} \sin(\omega t) = \sqrt{2} V_{\mathrm{rms}} \sin(\omega t)$$

Similarly for the current we have

$$I(t) = I_{\mathrm{peak}} \sin(\omega t - \varphi) = \sqrt{2} I_{\mathrm{rms}} \sin(\omega t - \varphi)$$

Note that they are not necessarily in phase, $\varphi \neq 0$.

The RMS values are useful because then for the **average power** with $\varphi = 0$ we can forget factors of 2

$$\langle P(t) \rangle = \langle V(t) I(t) \rangle = 2 V_{\mathrm{rms}} I_{\mathrm{rms}} \langle \sin^2(\omega t) \rangle = V_{\mathrm{rms}} I_{\mathrm{rms}}$$

Resistive loads

For purely **resistive loads**, e.g. a kettle or an electric heater, we have

$$V(t) = RI(t)$$

and thus for a voltage of $V(t) = \sqrt{2}V_{\mathrm{rms}}e^{j\omega t}$ (NB: for engineers $j = \sqrt{-1}$ to avoid confusion with the current i) we have

$$I(t) = \sqrt{2}\frac{V_{\mathrm{rms}}}{R}e^{j\omega t} = \frac{1}{R}V(t)$$

or in terms of the RMS value and phase shift

$$I_{\mathrm{rms}} = \frac{1}{R}V_{\mathrm{rms}}$$
$$\varphi = 0$$

Capacitive loads

For purely **capacitive loads** we have

$$I(t) = C\frac{dV(t)}{dt}$$

and thus for a voltage of $V(t) = \sqrt{2}V_{\mathrm{rms}}e^{j\omega t}$ we get

$$I(t) = \sqrt{2}j\omega CV_{\mathrm{rms}}e^{j\omega t} = j\omega CV(t)$$

or in terms of the RMS value and phase shift

$$I_{\mathrm{rms}} = \omega CV_{\mathrm{rms}}$$
$$\varphi = -\frac{\pi}{2}$$

We write $X_C = \frac{1}{\omega C}$ for the **capacitive reactance**.

Inductive loads

For purely **inductive loads**, e.g. a motor during start-up

$$V(t) = L\frac{dI(t)}{dt}$$

and thus for a voltage of $V(t) = \sqrt{2}V_{\mathrm{rms}}e^{j\omega t}$ we get

$$I(t) = \sqrt{2}\frac{V_{\mathrm{rms}}}{j\omega L}e^{j\omega t} = \frac{1}{j\omega L}V(t)$$

or in terms of the RMS value and phase shift

$$I_{\mathrm{rms}} = \frac{1}{\omega L}V_{\mathrm{rms}}$$
$$\varphi = \frac{\pi}{2}$$

We write $X_L = \omega L$ for the **inductive reactance**, in analogy to the resistance.

General loads

General loads will have a combination of resistive, capacitive and inductive parts. For an RLC circuit in series the voltage across the components is additive

$$V(t) = RI(t) + L\frac{dI(t)}{dt} + \frac{1}{C}\int_{-infty}^{t} I(\tau)d\tau$$

and therefore for a sinuisoidal voltage with angular frequency ω we get

$$V(t) = \left[R + j\omega L + \frac{1}{j\omega C}\right] I(t)$$

which leads us to define a general complex notion of resistance called **impedance**

$$Z = R + j\omega L + \frac{1}{j\omega C} = R + j(X_L - X_C) = R + jX$$

where X is the reactance $X = X_L - X_C$.

Impedances and admittances

Thus for a regular sinuisodal setup we have

$$V(t) = ZI(t)$$

where the complex **impedance** takes care both of the relation of the RMS values of the current and the voltage, and their phase difference. We can decompose Z into real resistance R and real reactance X

$$Z = R + jX$$

The inverse impedance, called the **admittance** is given by

$$Y = \frac{1}{Z}$$

so that

$$I(t) = YV(t)$$

We can also decompose this into real conductance G and real susceptance B

$$Y = G + jB$$

Simple transmission line

A simple model for a transmission line ℓ between nodes i and j is a resistance R in series with an (inductive) reactance X.

[Typical values are for a 380 kV overhead transmission line e.g. $R = 0.03$ Ohm/km and $X = 0.3$ Ohm/km.]

The voltage at each node (compared to ground) is given by $V_i(t) = \sqrt{2} V_i e^{j(\omega t + \theta_i)}$ where θ_i is the phase offset for each node and V_i is the RMS voltage magnitude.

Now the current in the transmission line is given by

$$I(t) = \frac{1}{R + jX} \left[V_j(t) - V_i(t) \right] = \frac{1}{R + jX} \sqrt{2} V_i e^{j(\omega t + \theta_i)} \left[\frac{V_j}{V_i} e^{j(\theta_j - \theta_i)} - 1 \right]$$

Active versus reactive power

Now let's consider the power injection at the first node. This is simply the voltage there multiplied by the current in the transmission line.

It's convenient to eliminate the time-dependent part $e^{j\omega t}$ by multiplying the voltage with the complex conjugate of the current

$$S = P + jQ = \frac{1}{2} V(t) I^*(t)$$

For a resistive load with $V(t) = RI(t)$ this reproduces the **active power** P.

For loads where the $I(t)$ is not in phase with the voltage, we get a flow of **reactive power** Q.

$S = P + jQ$ is called the **apparent power**.

Linearisation: Assumption 1/3

Now if we consider the power injected at the first node we get

$$P_i + jQ_i = \frac{1}{R + jX} V_i^2 \left[\frac{V_j}{V_i} e^{j(\theta_i - \theta_j)} - 1 \right]$$

This is the full non-linear equation for the power flow. Now let's linearise by making some simplifying assumptions.

1. Assume the voltage magnitudes are the same everywhere in the network $V_i = V_j$

$$P_i + jQ_i = \frac{1}{R + jX} V_i^2 \left[e^{j(\theta_i - \theta_j)} - 1 \right]$$

This means **power flows primarily according to angle differences** in this approximation.

Linearisation: Assumption 2/3

2. Now assume that the voltage angle differences across the transmission line are small enough that $\sin(\theta_i - \theta_j) \sim (\theta_i - \theta_j)$

$$
\begin{aligned}
P_i + jQ_i &= \frac{1}{R + jX} V_i^2 \left[e^{j(\theta_i - \theta_j)} - 1 \right] \\
&\sim \frac{1}{R + jX} V_i^2 \left[j(\theta_i - \theta_j) \right]
\end{aligned}
$$

This assumption is usually valid, since for stability reasons, we usually have in the transmission network $(\theta_i - \theta_j) \leq \frac{\pi}{6}$ (30 degrees).

Linearisation: Assumption 3/3

3. Finally we assume $R << X$ so that we can ignore the resistance R

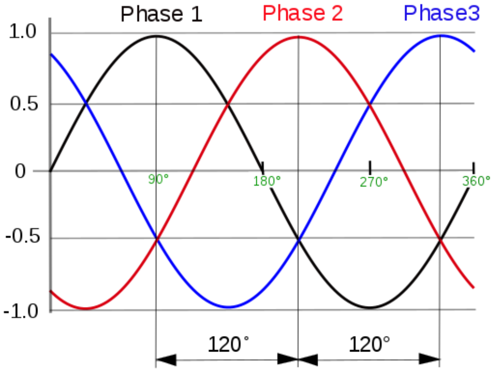
$$\begin{aligned} P_i + jQ_i &= \frac{1}{R + jX} V_i^2 \left[j(\theta_i - \theta_j) \right] \\ &\sim \frac{1}{jX} V_i^2 \left[j(\theta_i - \theta_j) \right] \\ &= \frac{V_i^2}{X} (\theta_i - \theta_j) \end{aligned}$$

Note that ignoring R means that we ignore resistive losses in the transmission lines and also since $Q_i \sim 0$, we ignore the flow of reactive power. Finally we absorb the voltage into the definition of the **per unit** reactance $x_\ell = \frac{X}{V_i^2}$ to get

$$f_\ell = P_i = -P_j = \frac{\theta_i - \theta_j}{x_\ell}$$

Three-phase power

Electricity is generally generated simultaneously in 3 separate circuits separate by 120 degrees or $\frac{2\pi}{3}$



Source: Wikipedia

In your plug, you only see one phase, but your oven may use all three phases.

Three-phase power

Why three phases? This was settled in the late 1880s.

1. The total power delivery is constant

$$\frac{d}{dt}P(t) = \frac{d}{dt}\left[P_a(t) + P_b(t) + P_c(t)\right] = 0$$

This reduces mechanical stress on generators and motors.

2. The sum of voltages and currents is zero, so no return path required! Saving on materials.

Both facts follow from

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi k}{N}} = 0$$

for $N > 1$.

3. Why $N = 3$ rather than $N = 2$? Allows directional rotating fields for induction motors (thanks Tesla!).

Rotating field in a three-phase induction motor

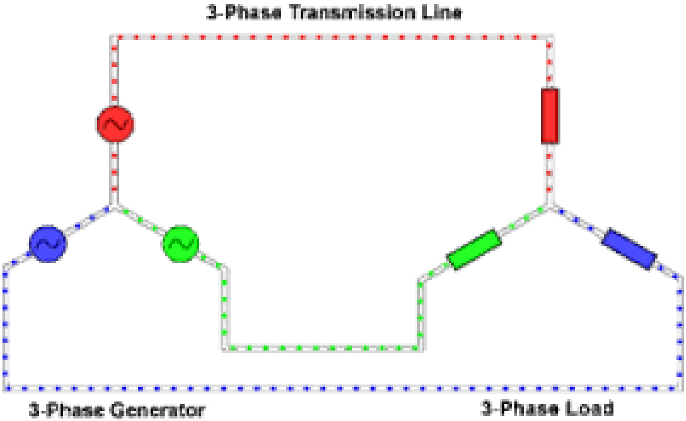
A brilliant insight (credited to Tesla, but the history is complicated) was that with three-phase power, you can place your wires spaced at $2\pi/3$ to create a **rotating** magnetic field

`https://www.youtube.com/watch?v=LtJoJBUSe28`

which can then induce a current in a rotor cage, which then experiences a torque thanks to the magnetic field: this is the principle of the **induction motor**.

It would not be possible to create such a rotating field with a single-phase or two-phase system.

Three-phase power



Source: Wikipedia

Computing the Linear Power Flow

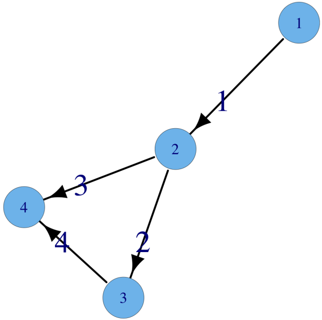
The goal of power flow analysis

The goal of a power/load flow analysis is to find the flows in the lines of a network given a power injection pattern at the nodes.
I.e. given power injection at the nodes

$$\mathbf{P}_i = \begin{pmatrix} 50 \\ 50 \\ 0 \\ -100 \end{pmatrix}$$

what are the flows in lines 1-4?

To find the flows, it is sufficient to know the **impedances** of the lines and the **voltages** at each node.



Framing the load flow problem

Suppose we have N nodes labelled by i, and L edges labelled by ℓ forming a directed graph G.

Suppose at each node we have a **power imbalance** p_i ($p_i > 0$ means its generating more than it consumes and $p_i < 0$ means it is consuming more than it).

Since we cannot create or destroy energy (and we're ignoring losses):

$$\sum_i p_i = 0$$

Question: How do the flows f_ℓ in the network relate to the nodal power imbalances?

Answer: According to the impedances (generalisation of resistance for oscillating voltage/current) and the corresponding voltages.

Kirchhoff's Current Law (KCL)

KCL says (in this linear setting) that the nodal power imbalance at node i is equal to the sum of direct flows arriving at the node. This can be expressed compactly with the incidence matrix

$$p_i = \sum_{\ell} K_{i\ell} f_{\ell} \qquad\qquad \forall i$$

Kirchhoff's Voltage Law (KVL)

KVL says that the sum of voltage differences across edges for any closed cycle must add up to zero.

If the voltage at any node is given by θ_i (this is infact the voltage **angle** - more next week) then the voltage difference across edge ℓ is

$$\sum_i K_{i\ell}\theta_i$$

And Kirchhoff's law can be expressed using the cycle matrix encoding of independent cycles

$$\sum_\ell C_{\ell c} \sum_i K_{i\ell}\theta_i = 0 \qquad\qquad \forall c$$

[Automatic, since we already said $KC = 0$.]

Kirchhoff's Voltage Law (KVL)

If we express the flow on each line in terms of the voltage angle (a relative of $V = IR$) then for a line ℓ with reactance x_ℓ

$$f_\ell = \frac{\theta_i - \theta_j}{x_\ell} = \frac{1}{x_\ell} \sum_i K_{i\ell} \theta_i$$

KVL now becomes

$$\sum_\ell C_{\ell c} x_\ell f_\ell = 0 \qquad \qquad \forall c$$

Solving the equations

If we combine

$$f_\ell = \frac{1}{x_\ell} \sum_i K_{i\ell} \theta_i$$

with Kirchhoff's Current Law we get

$$p_i = \sum_\ell K_{i\ell} f_\ell = \sum_\ell K_{i\ell} \frac{1}{x_\ell} \sum_j K_{j\ell} \theta_j$$

This is a **weighted Laplacian**. If we write $B_{k\ell}$ for the diagonal matrix with $B_{\ell\ell} = \frac{1}{x_\ell}$ then

$$L = KBK^t$$

and we get a **discrete Poisson equation** for the θ_i sourced by the p_i

$$p_i = \sum_j L_{ij} \theta_j$$

We can solve this for the θ_i and thus find the flows.

Solving the equations

Given p_i at every node, we want to find the flows f_ℓ. We have the equations

$$p_i = \sum_j L_{ij}\theta_j$$

$$f_\ell = \frac{1}{x_\ell}\sum_i K_{i\ell}\theta_i$$

Basic idea: invert L to get θ_i in terms of p_i

$$\theta_i = \sum_k (L^{-1})_{ik} p_k$$

then insert to get the flows as a linear function of the power injections p_i

$$f_\ell = \frac{1}{x_\ell}\sum_{i,k} K_{i\ell}(L^{-1})_{ik} p_k = \sum_k \mathrm{PTDF}_{\ell k} p_k$$

called the **Power Transfer Distribution Factors** (PTDF).

Inverting Laplacian L

There is one small catch: L is **not invertible** since we saw last time it has (for a connected network) one zero eigenvalue, with eigenvector $(1, 1, \dots 1)$, since by construction $\sum_j L_{ij} = 0$.

This is related to a gauge freedom to add a constant to all voltage angles

$$\theta_i \to \theta_i + c$$

which does not affect physical quantities:

$$p_i = \sum_j L_{ij}(\theta_j + c) = \sum_j L_{ij}(\theta_j)$$

$$f_\ell = \frac{1}{x_\ell} \sum_i K_{i\ell}(\theta_i + c) = \frac{1}{x_\ell} \sum_i K_{i\ell}(\theta_i)$$

Typically choose a **slack** or **reference bus** such that $\theta_1 = 0$.

Inverting Laplacian L

Two solutions:

1. Since $\theta_1 = 0$ and p_1 is not independent of the other power injections ($\sum_{i=1}^N p_i = 0$ implies $p_1 = -\sum_{i=2}^N p_i$), we can ignore these elements and invert the lower-right $(N-1) \times (N-1)$ part of L (which doesn't have zero eigenvalues) to find the remaining $\{\theta_i\}_{i=2,\dots N}$ in terms of the $\{p_i\}_{i=2,\dots N}$.

2. Use the Moore-Penrose pseudo-inverse.

Write L in terms of its basis of orthonormal eigenvectors e_i^n ($\sum_j L_{ij} e_j^n = \lambda_n e_i^n$, $\sum_i e_i^n e_i^n = 1$ and $\sum_i e_i^n e_i^m = 0$ if $n \neq m$):

$$L_{ij} = \sum_n \lambda_n e_i^n e_j^n$$

then the Moore-Penrose pseudo-inverse is:

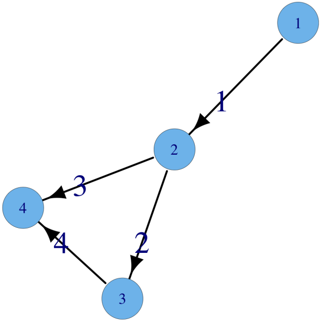
$$L_{ij}^\dagger = \sum_{n|\lambda_n \neq 0} \frac{1}{\lambda_n} e_i^n e_j^n$$

4-node example

$$\mathbf{K}_{i\ell} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

$$\mathbf{L}_{ij} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

$$\mathbf{PTDF}_{\ell i} = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -2/3 & -1/3 \\ 0 & 0 & -1/3 & -2/3 \\ 0 & 0 & 1/3 & -1/3 \end{pmatrix}$$

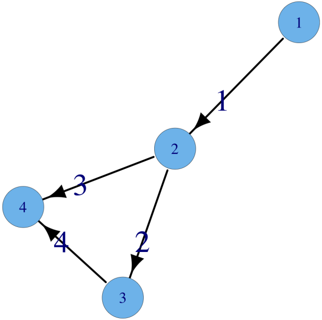


PTDF as sensitivity

Can also 'experimentally' determine the Power Transfer Distribution Factors (PTDF) by choosing a slack bus (in this case bus 1).

Each column (labelled by i) is then the resulting line flows if we have a simple power transfer from bus i to the slack $p_i = 1$ and $p_1 = -1$.

$$\mathbf{PTDF}_{\ell i} = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -2/3 & -1/3 \\ 0 & 0 & -1/3 & -2/3 \\ 0 & 0 & 1/3 & -1/3 \end{pmatrix}$$



Consequences of limiting power transfers

Line loading limits

You cannot pass infinite current through a transmission line.

As it warms, it sags, then it will become damaged and/or hit a building/tree and cause a short-circuit. For this reasons there are always **thermal limits** on current transfer. There may also be limits on the amount of power or current based on concerns about **voltage stability** or **general stability**.

Typically each line has a well-defined **line loading limit** on the amount of current or power that can flow through it:

$$|f_\ell| \leq F_\ell$$

where here F_ℓ is the maximum power capacity of the transmission line.

These limits prevent the transfer of renewable energy or other power sources.

Adjusting generator dispatch to avoid overloading

To avoid overloading the power lines, we must adjust our generator output (or the demand) so that the power imbalances do not overload the network.

We will now generalise and adjust our notation.

From lecture 2 we had for a single node:

$$-p_t = m_t - b_t + c_t = d_t - Ww_t - Ss_t - b_t + c_t = 0$$

where p_t was the nodal power balance, m_t was the mismatch (load d_t minus wind Ww_t and solar Ss_t), b_t was the backup power and c_t was curtailment.

We generalised this to multiple nodes labelled by i

$$-p_{i,t} = m_{i,t} - b_{i,t} + c_{i,t} = d_{i,t} - W_i w_{i,t} - S_i s_{i,t} - b_{i,t} + c_{i,t}$$

where now we don't enforce $p_{i,t} = 0$ but $\sum_i p_{i,t} = 0$ for all t.

Adjusting generator dispatch to avoid overloading

Now we write the dispatch of all generators at node i (wind, solar, backup) labelled by technology s as $g_{i,s,t}$ (i labels node, s technology and t time) so that we have a relation between load $d_{i,t}$, generation $g_{i,s,t}$ and network flows $f_{\ell,t}$

$$p_{i,t} = \sum_s g_{i,s,t} - d_{i,t} = \sum_\ell K_{i\ell} f_{\ell,t}$$

Where s runs over the wind, solar and backup capacity generators (e.g. hydro or natural gas) at the node.

A dispatchable generator's $g_{i,s,t}$ output can be controlled within the limits of its power capacity $G_{i,s}$

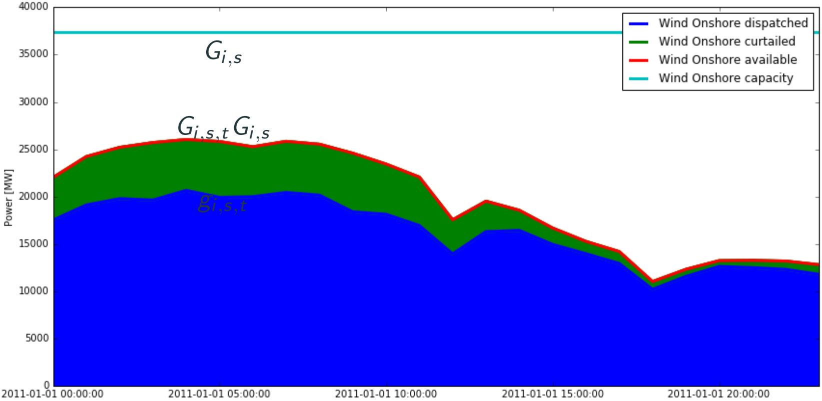
$$0 \leq g_{i,s,t} \leq G_{i,s}$$

Variable generation constraints

For a renewable generator we have time series of availability $0 \leq G_{i,s,t} \leq 1$ (the s_t and w_t before; W and S are the capacity $G_{i,s}$):

$$0 \leq g_{i,s,t} \leq G_{i,s,t} G_{i,s} \leq G_{i,s}$$

Curtailment corresponds to the case where $g_{i,s,t} < G_{i,s,t} G_{i,s}$:

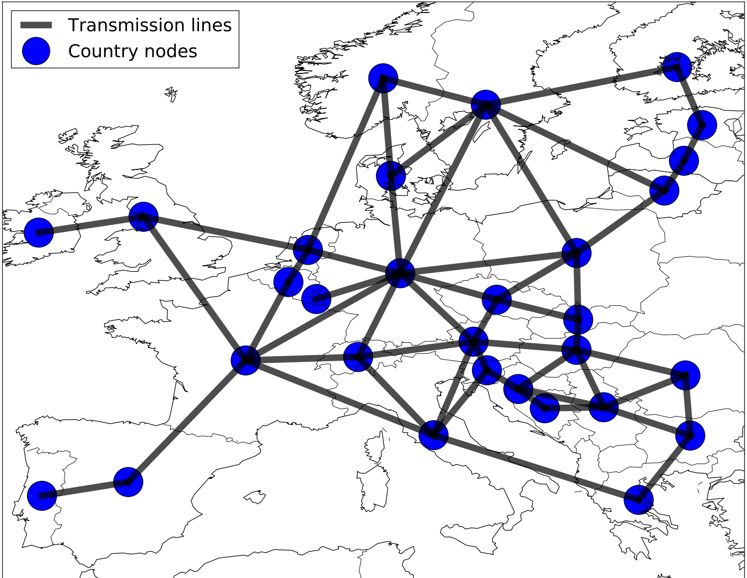


Germany curtailment example

See `https://pypsa.org/examples/scigrid-lopf-then-pf.html`.

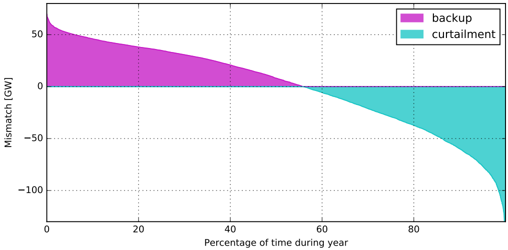
European transmission versus backup energy

Consider backup energy in a simplified European grid:

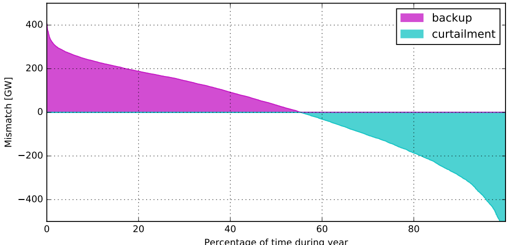


DE versus EU backup energy from last time

Germany needed backup generation for 31% of total load:



Europe needed Backup generation for only 24% of the total load:



European transmission versus backup energy

Transmission needs across a fully renewable European power system by Rodriguez, Becker, Andresen, Heide, Greiner, Renewable Energy, 2014