Energy System Modelling
Summer Semester 2020, Lecture 6

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Optimisation: Motivation

What to do about variable renewables?

Backup energy costs money and may also cause CO_2 emissions.

Curtailing renewable energy is also a waste.

We consider **four options** to deal with variable renewables:

1. Smoothing stochastic variations of renewable feed-in over **larger areas**, e.g. the whole of European continent.
2. Using **storage** to shift energy from times of surplus to deficit.
3. **Shifting demand** to different times, when renewables are abundant.
4. Consuming the electricity in **other sectors**, e.g. transport or heating.

Optimisation in energy networks is a tool to assess these options.

Why optimisation?

In the energy system we have lots of **degrees of freedom**:

1. Power plant and storage dispatch
2. Renewables curtailment
3. Dispatch of network elements (e.g. High Voltage Direct Current (HVDC) lines)
4. Capacities of everything when considering investment

but we also have to respect **physical constraints**:

1. Meet energy demand
2. Do not overload generators or storage
3. Do not overload network

and we want to do this while **minimising costs**. Solution: **optimisation**.

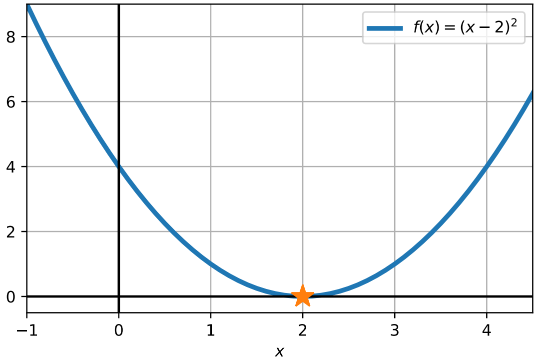
Optimisation: Introduction

Simplest 1-d optimisation problem

Consider the following problem. We have a function $f(x)$ of one variable $x \in \mathbb{R}$

$$f(x) = (x-2)^2$$

Where does it reach a minimum? School technique: find stationary point $\frac{df}{dx} = 2(x-2) = 0$, i.e. minimum at $x^* = 2$ where $f(x^*) = 0$.

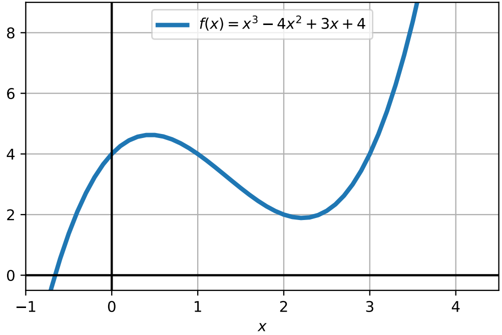


Simplest 1-d optimisation problem

Consider the following problem. We have a function $f(x)$ of one variable $x \in \mathbb{R}$

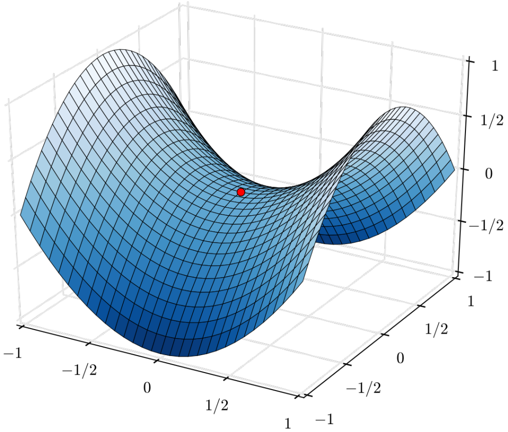
$$f(x) = x^3 - 4x^2 + 3x + 4$$

Where does it reach a minimum? School technique fails since has two stationary points, one local minimum and local maximum; must check 2nd derivative for minimum/maximum. Also: function is not bounded as $x \to -\infty$. No solution!



Beware saddle points in higher dimensions

Some functions have **saddle points** with zero derivative in all directions (stationary points) but that are neither maxima nor minima, e.g. $f(x,y) = x^2 - y^2$ at $(x,y) = (0,0)$.



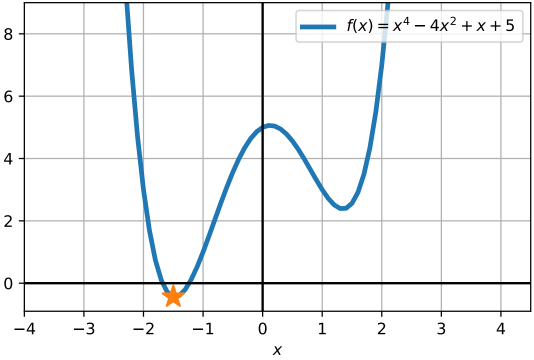
Source: Wikipedia

Simplest 1-d optimisation problem

Consider the following problem. We have a function $f(x)$ of one variable $x \in \mathbb{R}$

$$f(x) = x^4 - 4x^2 + x + 5$$

Where does it reach a minimum? Now two separate local minima. Function is **not convex** downward. This is a problem for algorithms that only search for minima locally.



Simplest 1-d optimisation problem with constraint

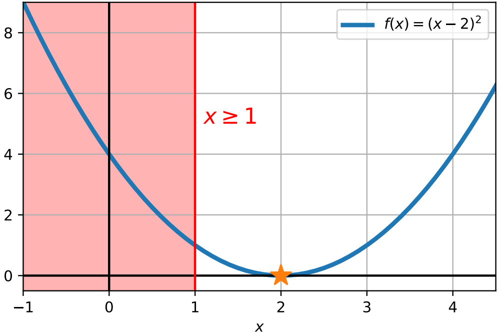
Consider the following problem. We minimise a function of one variable $x \in \mathbb{R}$

$$\min_x (x-2)^2$$

subject to a constraint

$$x \geq 1$$

The constraint has **no effect** on the solution. It is **non-binding**.



Simplest 1-d optimisation problem with constraint

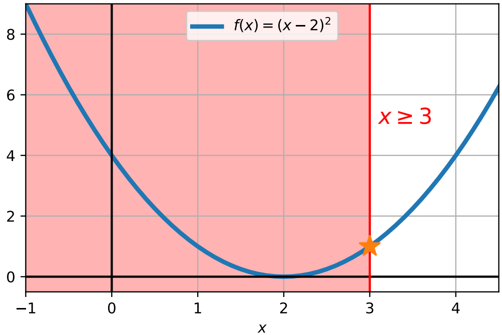
Consider the following problem. We minimise a function of one variable $x \in \mathbb{R}$

$$\min_x (x-2)^2$$

subject to a constraint

$$x \geq 3$$

Now the constraint is **binding** and is **saturated** at the optimum $x^* = 3$.



Simple 2-d optimisation problem

Consider the following problem. We have a function $f(x, y)$ of two variables $x, y \in \mathbb{R}$

$$f(x, y) = 3x$$

and we want to find the maximum of this function in the $x - y$ plane

$$\max_{x,y \in \mathbb{R}} f(x, y)$$

subject to the following constraints

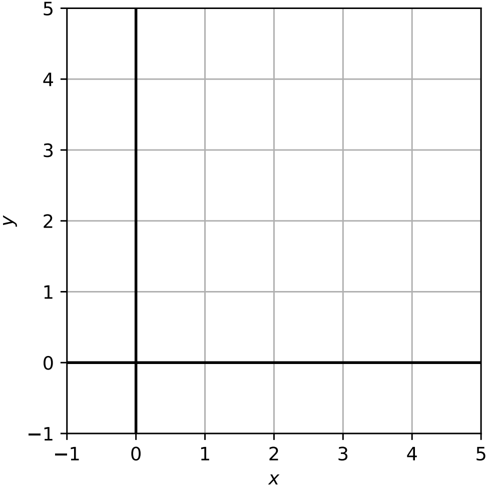
$$x + y \leq 4 \quad (1)$$

$$x \geq 0 \quad (2)$$

$$y \geq 1 \quad (3)$$

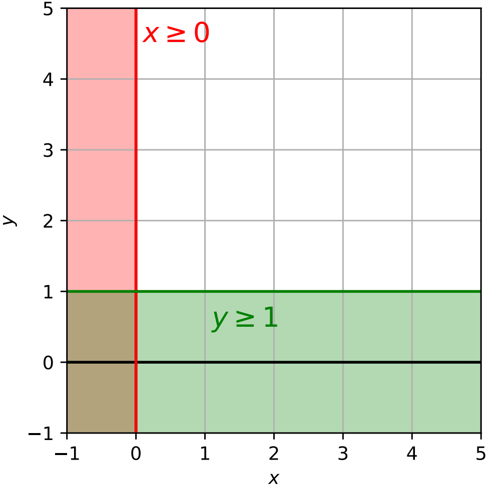
Simple 2-d optimisation problem

Consider $x - y$ plane of our variables:



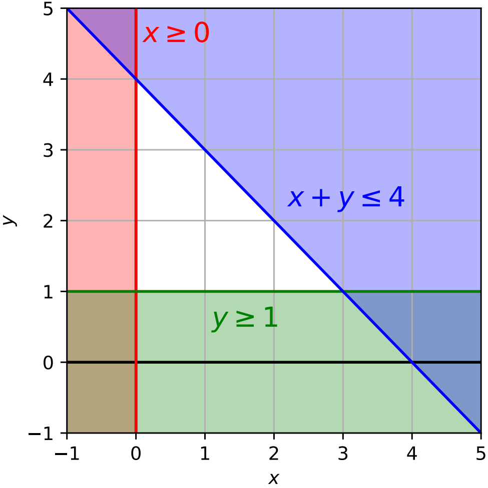
Simple 2-d optimisation problem

Add constraints (2) and (3):



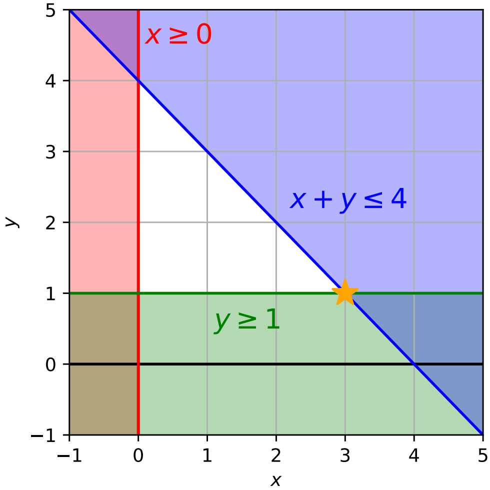
Simple 2-d optimisation problem

Add constraint (1). In this allowed space (white area) what is the maximum of $f(x, y) = 3x$?



Simple 2-d optimisation problem

$f(x, y) = 3x$ maximised at $x^* = 3, y^* = 1, f(x^*, y^*) = 9$:



Simple 2-d optimisation problem

Consider the following problem. We have a function $f(x, y)$ of two variables $x, y \in \mathbb{R}$

$$f(x, y) = 3x$$

and we want to find the maximum of this function in the $x - y$ plane

$$\max_{x,y \in \mathbb{R}} f(x, y)$$

subject to the following constraints

$$x + y \leq 4 \tag{4}$$

$$x \geq 0 \tag{5}$$

$$y \geq 1 \tag{6}$$

Optimal solution: $x^* = 3, y^* = 1, f(x^*, y^*) = 9.$

NB: We would have gotten the same solution if we had removed the 2nd constraint - it is **non-binding**.

Another simple optimisation problem

We can also have equality constraints. Consider the maximum of this function in the $x - y - z$ space

$$\max_{x,y,z \in \mathbb{R}} f(x, y, z) = (3x + 5z)$$

subject to the following constraints

$$\begin{aligned} x + y &\leq 4 \\ x &\geq 0 \\ y &\geq 1 \\ z &= 2 \end{aligned}$$

Another simple optimisation problem

We can also have equality constraints. Consider the maximum of this function in the $x - y - z$ space

$$\max_{x,y,z \in \mathbb{R}} f(x,y,z) = (3x + 5z)$$

subject to the following constraints

$$\begin{aligned} x + y &\leq 4 \\ x &\geq 0 \\ y &\geq 1 \\ z &= 2 \end{aligned}$$

Optimal solution: $x^* = 3, y^* = 1, z^* = 2, f(x^*, y^*, z^*) = 19.$

[This problem is **separable**: can solve for (x, y) and (z) separately.]

Energy system mapping to an optimisation problem

This optimisation problem has the same basic form as our energy system considerations:

Objective function to minimise	$\leftrightarrow$	**Minimise total costs**
Optimisation variables	$\leftrightarrow$	**Physical degrees of freedom (power plant dispatch, etc.)**
Constraints	$\leftrightarrow$	**Physical constraints (overloading, etc.)**

Before we apply optimisation to the energy system, we'll do some **theory**.

Optimisation: Theory

Optimisation problem

We have an **objective function** $f : \mathbb{R}^k \to \mathbb{R}$

$$\max_x f(x)$$

$[x = (x_1, \dots x_k)]$ subject to some **constraints** within $\mathbb{R}^k$:

$$\begin{array}{llll} g_i(x) = c_i & \leftrightarrow & \lambda_i & i = 1, \dots n \\ h_j(x) \leq d_j & \leftrightarrow & \mu_j & j = 1, \dots m \end{array}$$

λ_i and μ_j are the **Karush-Kuhn-Tucker (KKT) multipliers** (basically Lagrange multipliers) we introduce for each constraint equation. Each one measures the change in the objective value of the optimal solution obtained by relaxing the constraint by a small amount. Informally $\lambda_i \sim \frac{\partial f}{\partial c_i}$ and $\mu_j \sim \frac{\partial f}{\partial d_j}$ at the optimum x^*. They are also known as the **shadow prices** of the constraints.

Feasibility

The space $X \subset \mathbb{R}^k$ which satisfies

$$\begin{array}{cccc} g_i(x) = c_i & \leftrightarrow & \lambda_i & i = 1, \dots n \\ h_j(x) \leq d_j & \leftrightarrow & \mu_j & j = 1, \dots m \end{array}$$

is called the **feasible space**.

It will have dimension lower than k if there are independent equality constraints.

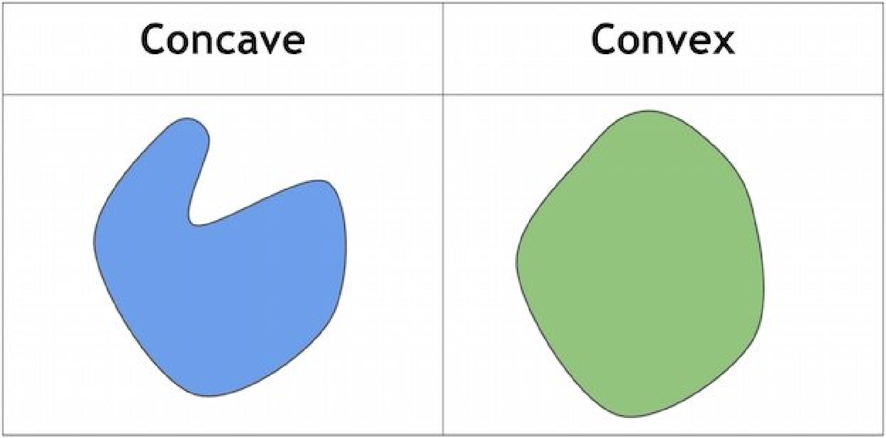
It may also be empty (e.g. for $k = 1$, $x \geq 1, x \leq 0$ in $\mathbb{R}^1$), in which case the optimisation problem is called **infeasible**.

It can be **convex** or **non-convex**.

If all the constraints are affine, then the feasible space is a convex polytope (multi-dimensional polygon).

Convexity means fast polynomial algorithms

If the feasible space is **convex** it is much easier to search, since for a convex objective function we can keep looking in the direction of improving objective function without worrying about getting stuck in a local maximum.



Lagrangian

We now study the **Lagrangian function**

$$\mathcal{L}(x, \lambda, \mu) = f(x) - \sum_i \lambda_i \left[g_i(x) - c_i\right] - \sum_j \mu_j \left[h_j(x) - d_j\right]$$

We've built this function using the variables λ_i and μ_j to better understand the optimal solution of $f(x)$ given the constraints.

The stationary points of $\mathcal{L}(x, \lambda, \mu)$ tell us important information about the optima of $f(x)$ given the constraints.

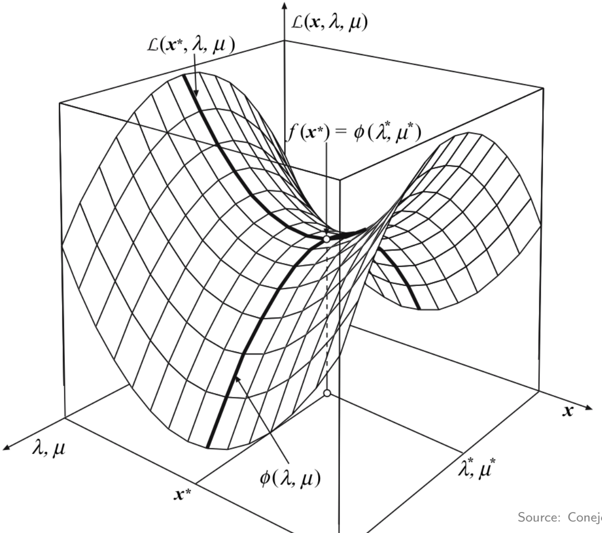
[It is entirely analogous to the physics Lagrangian $L(x, \dot{x}, \lambda)$ except we have no explicit time dependence $\dot{x}$ and we have additional constraints which are inequalities.]

We can already see that if $\frac{\partial \mathcal{L}}{\partial \lambda_i} = 0$ then the equality constraint $g_i(x) = c$ will be satisfied.

[Beware: $\pm$ signs appear differently in literature, but have been chosen here such that $\lambda_i = \frac{\partial \mathcal{L}}{\partial c_i}$ and $\mu_j = \frac{\partial \mathcal{L}}{\partial d_j}$.]

Optimum is a saddle point of the Lagrangian

The stationary point of $\mathcal{L}$ is a saddle point in (x, λ, μ) space (here minimising $f(x)$):



Source: Conejo et al, "Decomposition Techniques" (2006)

KKT conditions

The **Karush-Kuhn-Tucker (KKT) conditions** are necessary conditions that an optimal solution x^*, μ^*, λ^* always satisfies (up to some regularity conditions):

1. **Stationarity**: For $\ell = 1, \ldots k$
$$\frac{\partial \mathcal{L}}{\partial x_\ell} = \frac{\partial f}{\partial x_\ell} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial x_\ell} - \sum_j \mu_j^* \frac{\partial h_j}{\partial x_\ell} = 0$$

2. **Primal feasibility**:
$$g_i(x^*) = c_i$$
$$h_j(x^*) \leq d_j$$

3. **Dual feasibility**: $\mu_j^* \geq 0$

4. **Complementary slackness**: $\mu_j^*(h_j(x^*) - d_j) = 0$

Complementarity slackness for inequality constraints

We have for each inequality constraint

$$\mu_j^* \geq 0$$
$$\mu_j^*(h_j(x^*) - d_j) = 0$$

So **either** the inequality constraint is binding

$$h_j(x^*) = d_j$$

and we have $\mu_j^* \geq 0$.

Or the inequality constraint is NOT binding

$$h_j(x^*) < d_j$$

and we therefore MUST have $\mu_j^* = 0$.

If the inequality constraint is non-binding, we can remove it from the optimisation problem, since it has no effect on the optimal solution.

Nota Bene

1. The KKT conditions are necessary conditions for an optimal solution, but are only **sufficient** for optimality of the solution under certain conditions, e.g. for problems with convex objective, convex differentiable inequality constraints and affine equalities constraints. For linear problems, KKT is sufficient.
2. The variables x_ℓ are often called the **primary variables**, while (λ_i, μ_j) are the **dual variables**.
3. Since at the optimal solution we have $g_i(x^*) = c_i$ for equality constraints and $\mu_j^*(h_j(x^*) - d_j) = 0$, we have
$$\mathcal{L}(x^*, \lambda^*, \mu^*) = f(x^*)$$

How we will use the KKT conditions

Usually we will have enough constraints to determine the k values x_ℓ^* for $\ell = 1, \ldots k$ uniquely, i.e. k independent constraints will be binding and the objective function is never constant along any constraint.

We will use the KKT conditions, primarily stationarity, to determine the values of the k KKT multipliers for the independent binding constraints.

Dimensionality check: we need to find k KKT multipliers and we have k equations from stationarity to find them. Good!

The remaining KKT multipliers are either zero (for non-binding constraints) or dependent on the k independent KKT multipliers in the case of dependent constraints.

(There are also degenerate cases where the optimum is not at a single point, where things will be more complicated, e.g. when objective function is constant along a constraint.)

Return to simple optimisation problem

We want to find the maximum of this function in the $x - y$ plane

$$\max_{x,y \in \mathbb{R}} f(x,y) = 3x$$

subject to the following constraints (now with KKT multipliers)

$$\begin{aligned} x + y \leq 4 \quad &\leftrightarrow \quad \mu_1 \\ -x \leq 0 \quad &\leftrightarrow \quad \mu_2 \\ -y \leq -1 \quad &\leftrightarrow \quad \mu_3 \end{aligned}$$

We know the optimal solution in the **primal variables** $x^* = 3, y^* = 1, f(x^*, y^*) = 9$.

What about the **dual variables** μ_i?

Since the second constraint is not binding, by complementarity $\mu_2^*(-x^* - 0) = 0$ we have $\mu_2^* = 0$. To find μ_1^* and μ_3^* we have to do more work.

Simple problem with KKT conditions

We use stationarity for the optimal point:

$$0 = \frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial x} - \sum_j \mu_j^* \frac{\partial h_j}{\partial x} = 3 - \mu_1^* + \mu_2^*$$

$$0 = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial f}{\partial y} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial y} - \sum_j \mu_j^* \frac{\partial h_j}{\partial y} = -\mu_1^* + \mu_3^*$$

From which we find:

$$\mu_1^* = 3 - \mu_2^* = 3$$
$$\mu_3^* = \mu_1^* = 3$$

Check interpretation: $\mu_j = \frac{\partial \mathcal{L}}{\partial d_j}$ with $d_j \to d_j + \varepsilon$.

Simple problem with KKT conditions: Check interpretation

Check interpretation of $\mu_1^* = 3$ by shifting constant d_1 for first constraint by ε and solving:

$$\max_{x,y \in \mathbb{R}} f(x,y) = 3x$$

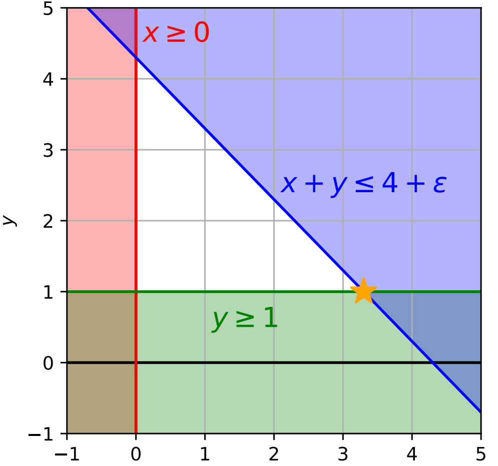
subject to the following constraints

$$x + y \leq 4 + \varepsilon \quad \leftrightarrow \quad \mu_1$$
$$-x \leq 0 \quad \leftrightarrow \quad \mu_2$$
$$-y \leq -1 \quad \leftrightarrow \quad \mu_3$$

Simple problem with KKT conditions: Check interpretation

$f(x, y) = 3x$ maximised at $x^* = 3 + \varepsilon, y^* = 1, f(x^*, y^*) = 9 + 3\varepsilon$.

$d_1 \to d_1 + \varepsilon$ causes optimum to shift $f(x^*, y^*) \to f(x^*, y^*) + 3\varepsilon$. Consistent with $\mu_1^* = 3$.



Return to another simple optimisation problem

We want to find the maximum of this function in the $x - y - z$ space

$$\max_{x,y,z\in\mathbb{R}} f(x,y) = 3x + 5z$$

subject to the following constraints (now with KKT multipliers)

$$\begin{aligned} x + y &\leq 4 && \leftrightarrow && \mu_1 \\ -x &\leq 0 && \leftrightarrow && \mu_2 \\ -y &\leq -1 && \leftrightarrow && \mu_3 \\ z &= 2 && \leftrightarrow && \lambda \end{aligned}$$

We know the optimal solution in the **primal variables**
$x^* = 3, y^* = 1, z^* = 2, f(x^*, y^*, z^*) = 19$.

What about the **dual variables** μ_i, λ?

We get same solutions to $\mu_1^* = 3, \mu_2^* = 0, \mu_3^* = 3$ because they're not coupled to z direction.
What about λ^*?

Another simple problem with KKT conditions

We use stationarity for the optimal point:

$$0 = \frac{\partial \mathcal{L}}{\partial z} = \frac{\partial f}{\partial z} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial z} - \sum_j \mu_j^* \frac{\partial h_j}{\partial z} = 5 - \lambda^*$$

From which we find:

$$\lambda^* = 5$$

Check interpretation: $\lambda_i = \frac{\partial \mathcal{L}}{\partial c_i}$ with $c_i \to c_i + \varepsilon$.

An example for you to do

Find the values of x^*, y^*, μ_i^*

$$\max_{x,y \in \mathbb{R}} f(x, y) = y$$

subject to the following constraints

$$\begin{aligned} y + x^2 &\leq 4 && \leftrightarrow && \mu_1 \\ y - 3x &\leq 0 && \leftrightarrow && \mu_2 \\ -y &\leq 0 && \leftrightarrow && \mu_3 \end{aligned}$$

Optimisation: Solution Algorithms

Optimisation solution algorithms

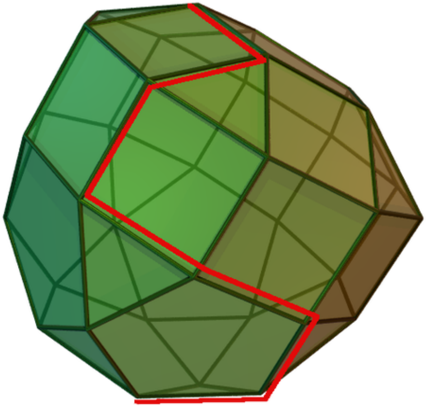
In general finding the solution to optimisation problems is hard, at worst NP-hard. Non-linear, non-convex and/or discrete (i.e. some variables can only take discrete values) problems are particularly troublesome.

There is specialised software for solving particular classes of problems (linear, quadratic, discrete etc.).

Since we will mostly focus on linear problems, the main two algorithms of relevance are:

- The **simplex algorithm**
- The **interior-point algorithm**

Simplex algorithm



The simplex algorithm works for linear problems by building the feasible space, which is a multi-dimensional polyhedron, and searching its surface for the solution.

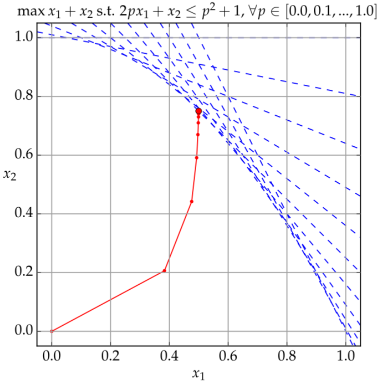
If the problem has a solution, the optimum can be assumed to always occur at (at least) one of the vertices of the polyhedron. There is a finite number of vertices.

The algorithm starts at a feasible vertex. If it's not the optimum, the objective function will increase along one of the edges leading away from the vertex. Follow that edge to the next vertex. Repeat until the optimum is found.

Complexity: On *average* over given set of problems can solve in polynomial time, but worst cases can always be found with exponential time.

Source: Wikipedia

Interior point methods



Interior point methods can be used on more general non-linear problems. They search the interior of the feasible space rather than its surface. They achieve this by extremising the objective function plus a **barrier term** that penalises solutions that come close to the boundary.
As the penality becomes less severe the algorithm converges to the optimum point at the boundary.

Complexity: For linear problems, Karmakar's version of the interior point method can run in polynomial time.

Source: Wikipedia

Interior point methods: Barrier method

Take a problem

$$\min_{\{x_i, i=1,\dots n\}} f(x)$$

such that for

$$c_j(x) = 0 \leftrightarrow \lambda_j, j = 1 \dots k$$
$$x \geq 0$$

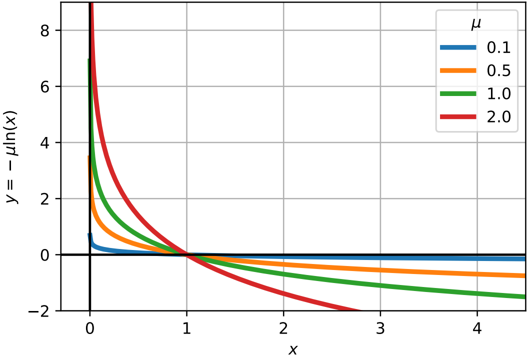
Any optimisation problem can be brought into this form. Introduce the **barrier function**

$$B(x, \mu) = f(x) - \mu \sum_{i=1}^{n} \ln(x_i)$$

where μ is the small and positive **barrier parameter** (a scalar). Note that the barrier term penalises solutions when x comes close to 0 by becoming large and positive.

Interior point methods: Barrier method

Barrier term $-\mu ln(x)$ penalises the minimisation the closer we get to $x = 0$. As μ gets smaller it converges on being a near-vertical function at $x = 0$.



Interior point methods: Barrier method: 1-d example

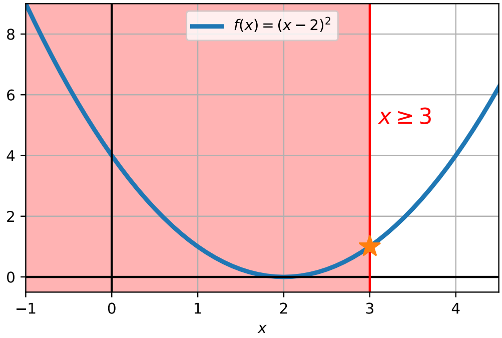
Return to our old 1-d example. We minimise a function of one variable $x \in \mathbb{R}$

$$\min_x (x-2)^2$$

subject to a constraint

$$x \geq 3$$

Solution: $x^* = 3$.

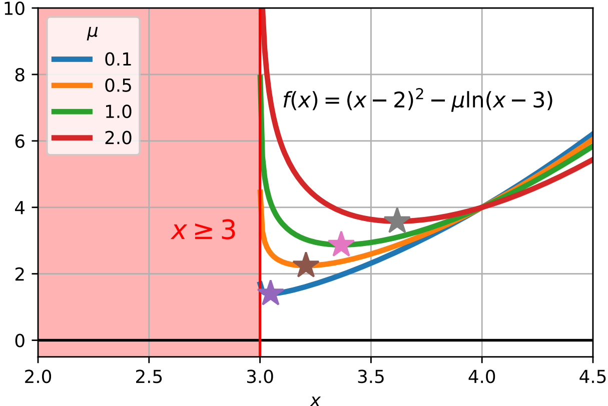


Interior point methods: Barrier method: 1-d example

Now instead minimise the barrier problem without any constraint:

$$\min_x B(x,\mu) = (x-2)^2 - \mu \ln(x-3)$$

Solve $\frac{\partial B(x,\mu)}{\partial x} = 2(x-2) - \frac{\mu}{x-3} = 0$, i.e. at $x^* = 2.5 + 0.5\sqrt{1+2\mu} \to 3$ as $\mu \to 0$.



Interior point methods: Barrier method

The problem

$$\min_{\{x_i, i=1,\dots n\}} \left[f(x) - \mu \sum_{i=1}^{n} \ln(x_i) \right]$$

such that

$$c_j(x) = 0 \leftrightarrow \lambda_j, j = 1 \dots k$$

can now be solved using the extremisation of the Lagrangian like we did for KKT sufficiency.

Solve the following equation system iteratively using the Newton method to find the x_i and λ_j:

$$\nabla_i f(x) - \mu \frac{1}{x_i} + \sum_j \lambda_j \nabla_i c_j(x) = 0$$

$$c_j(x) = 0$$

See this nice video for more details and visuals.