# AdS/CFT Beyond the Planar Limit

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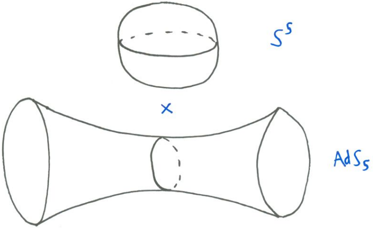
- Diagonal multi-matrix correlators and BPS operators in N=4 SYM (0711.0176 [hep-th]) TWB, Paul Heslop and Sanjaye Ramgoolam
- Permutations and the Loop (0801.2094 [hep-th]) TWB
- Diagonal free field matrix correlators, global symmetries and giant gravitons (0806.1911 [hep-th]) TWB, PJH, SR
- Thesis and unpublished

# IIB superstrings on $AdS_5 \times S^5$



$$\frac{1}{\alpha'} \int d\tau d\sigma \, [\text{ non-linear } \sigma\text{-model }]$$

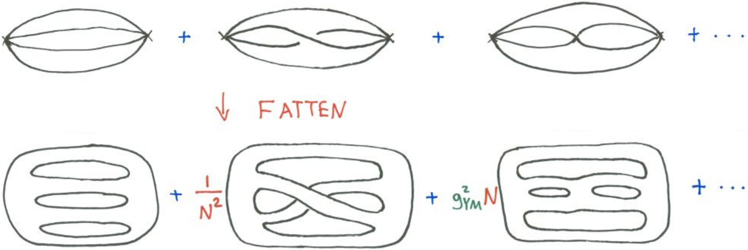
Perturbative expansion in the string coupling $g_s$

# $\mathcal{N} = 4$ SUSY Yang-Mills: a Conformal Field Theory

$$\frac{N}{\lambda}\int d^4x\,\mathrm{tr}\Big[F_{\mu\nu}F^{\mu\nu} + D^{\mu}\phi_i D_{\mu}\phi_i - [\phi_i,\phi_j][\phi_i,\phi_j]$$

$$+\psi\sigma^{\mu}D_{\mu}\psi - \psi\phi\psi\Big] \;+\; \theta\int d^4x\,\mathrm{tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}]$$

Gauge group $U(N)$; fields in adjoint. Compute correlation functions of gauge-invariant local operators.



# The AdS/CFT correspondence

$$\left\{ \begin{array}{c} \text{IIB superstrings on} \\ AdS_5 \times S^5 \end{array} \right\} = \left\{ \begin{array}{c} \mathcal{N} = 4 \text{ SUSY} \\ \text{Yang-Mills in 4d} \end{array} \right\}$$

$$\frac{R^2}{\alpha'} = \sqrt{\lambda}$$

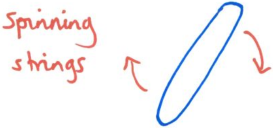
$$g_s = \frac{\lambda}{N}$$

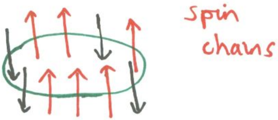
bosonic symmetries $SO(2,4) \times SO(6)$ match

# The Planar Limit

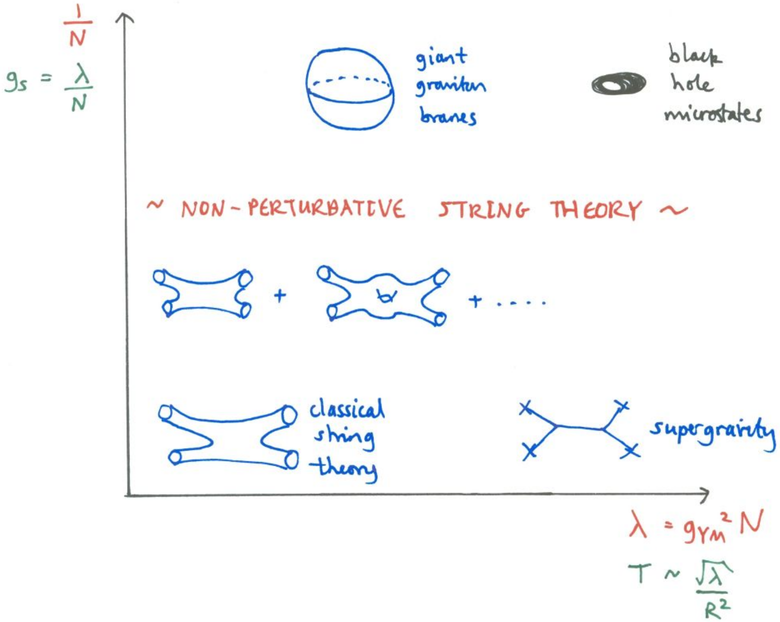
AdS/CFT has been successfully studied in the planar limit.
$\lambda$ fixed, $N \to \infty \Rightarrow g_s \to 0$.

- Single-trace local operators in $\mathcal{N} = 4$ SYM.
- Classical string theory in bulk for strict $N \to \infty$ limit.
- Beautiful story of spinning strings, spin chains and integrability.





# Parameter space



# General Programme

We want to study AdS/CFT at finite N.

- Multi-trace and determinant-type operators in CFT.
- Must deal with Stringy Exclusion Principle.
- Correlation functions involve complicated combinatorics.
- Non-perturbative quantum gravity effects in the bulk, giant graviton branes, black holes.

# Beasts of the field

In $\mathcal{N} = 4$ super Yang-Mills the fields are

$$X, Y, Z, X^\dagger, Y^\dagger, Z^\dagger; \lambda^A_\alpha, \bar{\lambda}^A_{\dot{\alpha}}; F_{\mu\nu} \quad \text{plus derivatives} \quad D_\mu$$

Each field is in the adjoint of the gauge group $U(N)$

$$(W_m)^i_j$$

$i, j = 1, 2 \ldots N$.
$m$ runs over different fields.

To get gauge-invariant operators, usual route is to multiply these $N \times N$ matrices together and take traces

$$:\mathrm{tr}(XYX^\dagger)\,\mathrm{tr}(YZ)\,\mathrm{tr}(Y^\dagger): \;=\; X^{i_1}_{i_2}\,Y^{i_2}_{i_3}\,X^{\dagger\,i_3}_{\;\;i_1}\;\;Y^{i_4}_{i_5}\,Z^{i_5}_{i_4}\;\;Y^{\dagger\,i_6}_{\;\;i_6}\;-\;\ldots$$

# Correlation functions

Wick contract with, e.g.,

$$\left\langle X^i_j(x)\;\; X^{\dagger k}{}_l(y) \right\rangle \;\; = \;\; \delta^i_l \delta^k_j \, \frac{1}{(x-y)^2}$$

Even at tree level this gives a complicated $\frac{1}{N}$ expansion

$$\left\langle \mathrm{tr}(XXXX)[x]\;\; \mathrm{tr}(X^\dagger X^\dagger X^\dagger X^\dagger)[y] \right\rangle = \frac{4N^4}{(x-y)^8}\; \left(1 + \frac{5}{N^2}\right)$$

$$\left\langle \mathrm{tr}(XXXX)[x]\;\; \mathrm{tr}(X^\dagger X^\dagger)\,\mathrm{tr}(X^\dagger X^\dagger)[y] \right\rangle = \frac{4N^4}{(x-y)^8} \left(\frac{4}{N} + \frac{2}{N^3}\right)$$

Mixing between different trace structures only suppressed when the op. length $n < N$. [For giant graviton $n \sim N$, black hole $n \sim N^2$.]

# Stringy Exclusion Principle

For an $N \times N$ matrix $A$, traces of powers bigger than $N$ can always be written in terms of traces of powers $\leq N$.

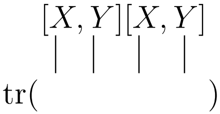
For example, if $N = 2$ for the $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\mathrm{tr}(A^3) = \frac{3}{2}\,\mathrm{tr}(A^2)\,\mathrm{tr}(A) - \frac{1}{2}\,\mathrm{tr}(A)\,\mathrm{tr}(A)\,\mathrm{tr}(A)$$

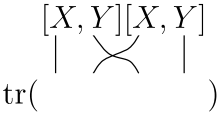
So working with traces is problematic for operators with $\Delta \geq N$ ...

# Operators with multiple fields

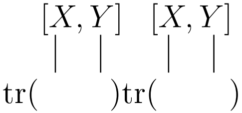
Trace the same field content (e.g. for $U(2)$ rep $\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}$) and you get



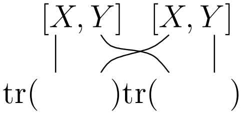
$$= \operatorname{tr}([X,Y][X,Y])$$



$$= 0$$



$$= 0$$



$$= \operatorname{tr}(\Phi^r\Phi^s)\operatorname{tr}(\Phi_r\Phi_s)$$

where $\Phi^p\Phi_p = \epsilon^{pq}\Phi_p\Phi_q = [X,Y]$.

# Solution: separation

Separate:

- Representation of global symmetry group $PSU(2,2|4)$, which organises field content and its symmetrisation

from

- Trace structure, which it turns out will involve the representation theory of the gauge group $U(N)$

The permutation group $S_n$ plays a vital role.

Simplest to see for half-BPS case, where the representation of the global symmetry group is trivial.

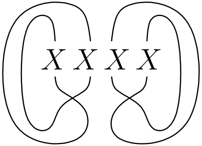
# Half BPS operators: only trace structure

Here we have only one type of field: $X$. Multi-trace ops labelled by (conjugacy classes of) elements of the symmetric group $S_n$.

E.g. $\mathrm{tr}(XX)\,\mathrm{tr}(XX)$ can be written using $\alpha = (12)(34) \in S_4$

$$\mathrm{tr}(XX)\,\mathrm{tr}(XX) = X^{i_1}_{i_{\alpha(1)}}\ X^{i_2}_{i_{\alpha(2)}}\ X^{i_3}_{i_{\alpha(3)}}\ X^{i_4}_{i_{\alpha(4)}} = X^{i_1}_{i_2}\ X^{i_2}_{i_1}\ X^{i_3}_{i_4}\ X^{i_4}_{i_3}$$

$\mathrm{tr}(\alpha X^{\otimes 4}) \equiv$



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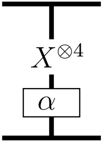


# The Schur polynomials

Define linear change of basis to Schur polynomials

$$\mathrm{tr}_R(X^{\otimes n}) \equiv \frac{1}{n!}\sum_{\alpha\in S_n}\chi_R(\alpha)\;\; X^{i_1}_{i_{\alpha(1)}} X^{i_2}_{i_{\alpha(2)}}\cdots X^{i_n}_{i_{\alpha(n)}}$$

$R$ is Young diagram of $n$ boxes: rep both of $U(N)$ and $S_n$, sorts multi-trace structure (cf. Wilson loop). 2-pt function diagonal

$$\left\langle \chi_R(X(x))\,\chi_S(X^{\dagger}(y))\right\rangle = \;\delta_{RS}\;\; \mathrm{Dim}_N\;R$$

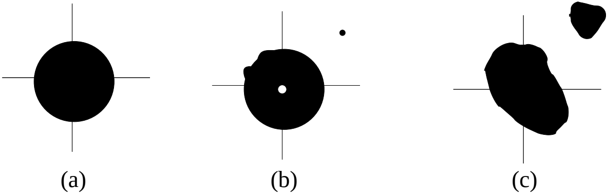
$\mathrm{Dim}_N\;R$ is $U(N)$ dimension of $R$; it capture the $N$ expansion, e.g.

$$Dim_N \ydiagram{3,2,1} = \frac{N^2(N+1)(N+2)(N-1)N(N-2)}{45}$$

(The half-BPS sector is not renormalised, so this holds for all values of the coupling $\lambda$. This will not be true in general...)

# Physical meaning of Schur polynomials

- Encode finite $N$ stringy exclusion principle, since reps of $U(N)$ have Young diagrams with at most $N$ rows.
- Row-lengths $\sim N$ occupied energy levels of free fermions from complex matrix model.
- For $n \sim N$ map to giant gravitons, single column $[1^N]$ to giant in $S^5$, single row $[N]$ in $AdS_5$. General: LLM-type geometries.
- Can gain qualitative understanding of black hole microstates.



# Outline of method for multiple non-commuting fields

Solution: group theory.

Organise multi-trace operators of $\mathcal{N} = 4$ SYM at finite $N$ into reps of the global symmetry group and reps of the local gauge group (which will control multi-trace structure à la Wilson loop).

1. Start with $n$ fields with none of their indices contracted

$$(W_{m_1})^{i_1}_{j_1}\ (W_{m_2})^{i_2}_{j_2}\ \cdots\ (W_{m_n})^{i_n}_{j_n}$$

2. Build into reps of $G$ and $U(N)$.
3. Enforce gauge invariance.

# Technical slide 1/4: Example of $U(2)$

Take the fundamental representation $V_F$ of $U(2)$

$$V_F = \begin{pmatrix} X \\ Y \end{pmatrix}$$

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Take the fundamental representation $V_F$ of $U(2)$

$$V_F = \begin{pmatrix} X \\ Y \end{pmatrix}$$

and then consider the simplest tensor product

$$V_F \otimes V_F$$

We can re-arrange into irreducible reps of $U(2)$

$$\begin{pmatrix} X \\ Y \end{pmatrix} \otimes \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \otimes X \\ X \otimes Y + Y \otimes X \\ Y \otimes Y \end{pmatrix} \oplus \begin{pmatrix} X \otimes Y - Y \otimes X \end{pmatrix}$$

$$\square \quad \otimes \quad \square \quad = \quad \square\!\square \quad \oplus \quad \begin{array}{c} \square \\[-4pt] \square \end{array}$$

# Technical slide 2/4

We hit a problem with three copies of the fundamental

$$V_F^{\otimes 3} = \begin{pmatrix} X \otimes X \otimes X \\ \vdots \end{pmatrix} \oplus \begin{pmatrix} X \otimes X \otimes {\color{red}Y} - X \otimes {\color{red}Y} \otimes X \\ \vdots \end{pmatrix}$$

$$\oplus \begin{pmatrix} X \otimes X \otimes {\color{red}Y} - {\color{red}Y} \otimes X \otimes X \\ \vdots \end{pmatrix}$$

In terms of Young diagrams

$$\ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} = \ydiagram{3} \oplus {\color{red}2}\, \ydiagram{2,1}$$

How do we account for this multiplicity?

# Technical slide 3/4: Schur-Weyl duality

For $V_F^{\otimes n}$, an $n$-tensor products of the fundamental of $U(K)$, there are two commuting group actions:

- $U(K)$: the action of $U(K)$ on its fundamental rep
- $S_n$: permutes the $n$ different copies of $V_F$

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- $S_n$: permutes the $n$ different copies of $V_F$

So organise $V_F^{\otimes n}$ in terms of representations of the two groups:

$$V_F^{\otimes n} \equiv \overbrace{\square \otimes \square \otimes \cdots \otimes \square}^{n} = \bigoplus_{\Lambda} V_\Lambda^{U(K)} \otimes V_\Lambda^{S_n}$$

where $\Lambda$ runs over Young diagrams with $n$ boxes and at most $K$ rows.

[To answer question: dim $V^{S_3}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = 2$. ]

# Technical slide 4/4: Clebsch-Gordan coefficients

We can express this map from $V_F^{\otimes n}$ to reps of $U(K)$ and $S_n$ using Clebsch-Gordan coefficients $C$.

$$C : \qquad V_F^{\otimes n} \qquad \to \quad V_\Lambda^{U(K)} \otimes V_\Lambda^{S_n}$$

$$C^{m_1 m_2 \ldots m_n}_{\Lambda, M_\Lambda, a_\Lambda} \quad W_{m_1} \otimes W_{m_2} \otimes \cdots \otimes W_{m_n} \quad = \quad |\Lambda, M_\Lambda, a_\Lambda\rangle$$

- $m_k = \{1, 2, \ldots K\}$ (for $U(2)$, $W_1 = X$, $W_2 = Y$)
- $M_\Lambda$ labels $U(K)$ state in $V_\Lambda^{U(K)}$
- $a_\Lambda$ labels $S_n$ state in $V_\Lambda^{S_n}$
- Clebsch map is invertible

# The solution: use C-G coefficients

Consider operators with $n$ fields, a generic example being

$$(W_{m_1})^{i_1}_{j_1}\,(W_{m_2})^{i_2}_{j_2}\,\cdots\,(W_{m_n})^{i_n}_{j_n}$$

where $\{W_m\}$ are the fields of a subsector $G \subset PSU(2,2|4)$.

Combine indices into rep of $G \times S_n$ and two of $U(N) \times S_n$

$$C^{m_1 \dots m_n}_{\Lambda(G),M_\Lambda,a_\Lambda}\; C^{i_1 \dots i_n}_{R(U(N)),M_R,a_R}\; C^{j_1 \dots j_n}_{\bar{S}(U(N)),M_S,a_S}\; (W_{m_1})^{i_1}_{j_1} \cdots (W_{m_n})^{i_n}_{j_n}$$

- Enforce gauge invariance: pick singlet $\mathbf{1} \in R \otimes \bar{S}$ (implies $R = S$, sum over $M_R = M_S$)
- Impose overall $S_n$ invariance

# Operator for general $G$

$$\mathcal{O}\Big[\overbrace{\Lambda\,(G),\ M_{\Lambda}}^{\text{rep and state of } G},\ \underset{\substack{\nearrow \\ R \text{ of } U(N) \text{ gives multi-trace structure}}}{R\,(U(N))},\ \underset{\substack{\nwarrow \\ \text{(multiplicity)}}}{\tau}\Big]$$

- Complete basis on space of multi-trace operators at finite $N$ built out of fundamental fields of $G$.
- Free 2-point function totally diagonal on all labels, proportional to $\mathrm{Dim}_N\ R$.
- Operators given in detail for $G = U(3), SL(2), O(2), SO(2,4)$, prescription given for $SO(6)$.
- For $SL(2)$, in regime of large quantum numbers, spectrum of our basis matches excitations of giant gravitons.

# Partition function at finite $N$

At finite $N$ we have for the free theory

$$\mathcal{Z}_N = \int_{U(N)} [dU] \ \exp\left\{ \sum_{m=1}^{\infty} \frac{1}{m} f(\mathbf{x}^m) \mathrm{tr}(U^\dagger)^m \mathrm{tr} U^m \right\}$$

where $f(\mathbf{x})$ is the character for the fundamental fields; for $U(K)$ this is just the trace of the matrix

$$f(\mathbf{x}) = \chi_F^{U(K)}(\mathbf{x}) = x_1 + x_2 + \cdots + x_K$$

Expanding we match exactly the coefficient of each irrep of $G$

$$\mathcal{Z}_N = \sum_n \sum_{\Lambda(U(K))} \sum_{R(U(N))} C(R, R, \Lambda) \ \ \chi_\Lambda(\mathbf{x})$$

where $C(R, R, \Lambda)$ is the number of times $\Lambda$ appears in the symmetric group tensor product $R \otimes R$.

# Subsectors

We can do this classification for the following sub-sectors $G \subset PSU(2,2|4)$ of the global superconformal symmetry group (and product groups $G_1 \times G_2$) :

$$
\begin{aligned}
\text{half BPS } U(1): \quad \{W_m\} \quad &= \{X\} \\
U(3): \quad \{W_m\} \quad &= \{X, Y, Z\} \\
U(3|2): \quad \{W_m\} \quad &= \{X, Y, Z; \psi_1, \psi_2\} \\
O(2): \quad \{W_m\} \quad &= \{X, X^\dagger\} \\
SO(6): \quad \{W_m\} \quad &= \{X, Y, Z, X^\dagger, Y^\dagger, Z^\dagger\} \\
SL(2,\mathbb{R}): \quad \{W_m\} \quad &= \{X, \partial X, \partial^2 X, \partial^3 X, \ldots\} \\
SO(2,4): \quad \{W_m\} \quad &= \{X, \partial_\mu X, \partial_\mu \partial_\nu X, \ldots\}
\end{aligned}
$$

# More complicated example: $SL(2,\mathbb{R})$

Take arbitrarily many derivatives of a field $\{W_m\} = \{\partial^m X\}$

$$\partial^{m_1} X \otimes \partial^{m_2} X \otimes \cdots \otimes \partial^{m_n} X$$

In $V_F^{\otimes n}$ sort into ops with $k = m_1 + \cdots + m_n$ derivatives (spread out across the $n$ sites) and remove all descendants of form $D^p(\cdots)$

$$\begin{tikzcd}
 & D = \partial^{(1)} + \partial^{(2)} + \cdots + \partial^{(n)} \\
\begin{pmatrix} \partial^{(1)} \\ \partial^{(2)} \\ \vdots \\ \partial^{(n)} \end{pmatrix} \arrow[ur] \arrow[dr] & \\
 & \begin{pmatrix} \partial^{(1)} - \partial^{(2)} \\ \partial^{(2)} - \partial^{(3)} \\ \vdots \\ \partial^{(n-1)} - \partial^{(n)} \end{pmatrix}
\end{tikzcd}$$

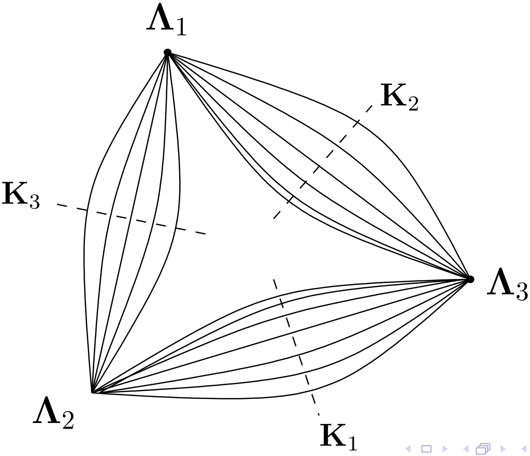
This is split of the canonical permutation rep $V_{\mathrm{nat}}$ of $S_n$ into the trivial and the 'standard' rep $V_{\mathrm{nat}} = V_{[n]} \oplus V_{[n-1,1]}$.

Build HWS of $SL(2,\mathbb{R})$ with $V_{[n-1,1]}$ (i.e. the differences).

# Free three-point function

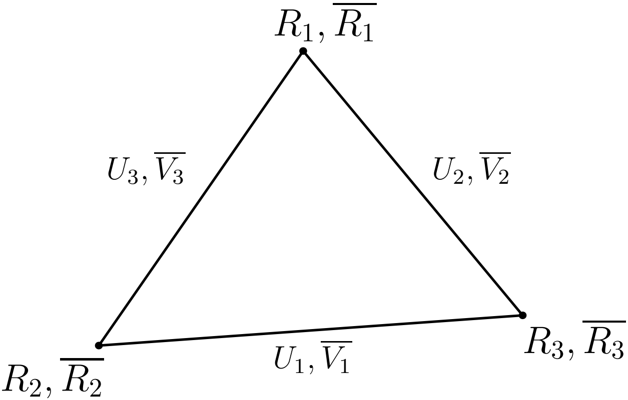
We can also use this formalism to work out the free non-extremal three-point function

$$\left\langle \mathcal{O}[\mathbf{\Lambda}_1, R_1](x_1)\;\; \mathcal{O}[\mathbf{\Lambda}_2, R_2](x_2)\;\; \mathcal{O}[\mathbf{\Lambda}_3, R_3](x_3) \right\rangle$$



# Three-point gauge spin network

On the legs between the operators the gauge group representations need not form a singlet. The three-point function becomes a $G \times U(N)$ spin network. Also extends to one-loop...



# One loop two-point function

At one loop this basis is no longer diagonal. Operators mix and we must rediagonalise. Multiplets also re-organise in a highly non-trivial way. Take for example the $U(2)$ sector, $\Lambda = \begin{array}{|c|c|}\hline \phantom{x} & \phantom{x} \\ \hline \phantom{x} & \phantom{x} \\ \hline \end{array}$.

Some operators are no longer protected and join long multiplets

$$\mathrm{tr}([X, Y][X, Y]) \qquad \Delta = 4 + \frac{3\lambda}{4\pi^2} + \mathcal{O}(\lambda^2)$$

(This is a descendant of the Konishi at weak coupling.)

The $\frac{1}{4}$-BPS operators, which are protected, are in 1-to-1 correspondence with the chiral ring and receive $\frac{1}{N}$ corrections, e.g.

$$\mathrm{tr}(\Phi^r \Phi^s)\,\mathrm{tr}(\Phi_r \Phi_s) + \frac{2}{N}\,\mathrm{tr}([X, Y][X, Y])$$

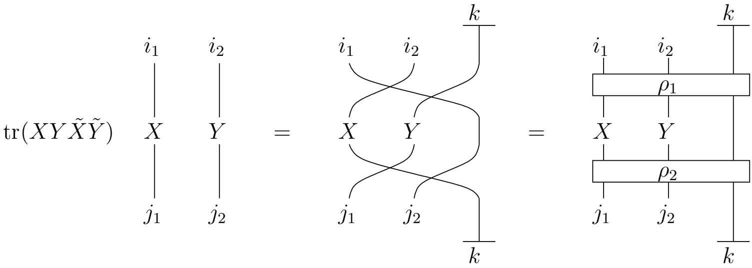
# Action of dilatation operator

Analyse mixing with one-loop dilatation operator, e.g. $U(2)$ sector

$$:\mathrm{tr}([X,Y][\tilde{X},\tilde{Y}]):$$

$\tilde{X} \sim \frac{\partial}{\partial X}$. This gives matrix of anomalous dimensions.

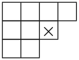
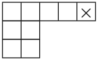
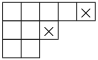
Write its action on two sites by introducing extra $(n+1)$th index

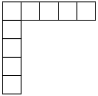


# Constrained mixing at one loop

Operators now mix via $(n+1)$-box $U(N)$ reps $T$.

The $U(N)$ $n$-box representations $R$ and $S$, controlling multi-trace structure, then only mix if they both fit into the same $(n+1)$-box rep $T$, i.e. $R$ and $S$ must be related by repositioning a single box.

$R =$  mixes with $S_1 =$  via $T =$ 

but not with $S_2 =$ 

# Solution for commuting matrices

$\frac{1}{4}$ and $\frac{1}{8}$-BPS ops at weak coupling in chiral ring built from symmetrised traces, i.e. commuting matrices. Characterise in terms of symmetric functions of eigenvalues. Still transform under $S_N \subset U(N)$; want invariants of this group from $\left(V_{\mathrm{nat}}^{S_N}\right)^{\otimes n}$.

$$C^{m_1 \ldots m_n}_{\Lambda(G), M_\Lambda, a_\Lambda} \; C^{e_1 \ldots e_n}_{[N](S_N), \Lambda(S_n), a_\Lambda} \quad x^{e_1}_{m_1} \; x^{e_2}_{m_2} \; \cdots \; x^{e_n}_{m_n}$$

$e_i \in \{1, 2, \ldots N\}$. Generating function for multiplicity at finite $N$

$$\prod_{m,n=0}^{\infty} \frac{1}{1 - \nu x^m y^n} = \sum_{N=0}^{\infty} \nu^N \mathcal{Z}_N^{\frac{1}{4}\text{-}BPS}(x, y)$$

$$\mathcal{Z}_N^{\frac{1}{4}\text{-}BPS}(x, y) = \sum_{\Lambda \text{ of } U(2)} \mathrm{dim}^{S_N \times S_n}_{[N], \Lambda} \; \chi_\Lambda(x, y)$$

Map combinatorics to supergravity geometries à la LLM?

# Conclusions

- To study many phenomena in AdS/CFT need $N$ finite.
- Organised multi-trace ops into complete basis that transforms in irreps of $G \subset PSU(2,2|4)$, traces sorted by $U(N)$.
- This basis diagonalises the free two-point function, including all finite $N$ corrections. Higher-point functions also simple.
- One-loop mixing highly constrained.
- $\frac{1}{4}$ and $\frac{1}{8}$-BPS ops in chiral ring characterised in terms of functions of eigenvalues of fields.
- Focus in future:
  - Clarify field versus bulk description of $\frac{1}{4}$ and $\frac{1}{8}$-BPS states.
  - String theory dual to zero/weakly-coupled field theory.
  - Diagonalise spectrum at 1-loop.
  - Sixteenth-BPS states: how do they furnish black hole entropy?
  - Understand information loss.