AdS/CFT Beyond the Planar Limit

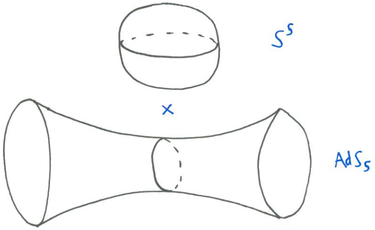
T.W. Brown

Queen Mary, University of London

Durham, October 2008

- Diagonal multi-matrix correlators and BPS operators in N=4 SYM (0711.0176 [hep-th]) TWB, Paul Heslop and Sanjaye Ramgoolam
- Permutations and the Loop (0801.2094 [hep-th]) TWB
- Diagonal free field matrix correlators, global symmetries and giant gravitons (0806.1911 [hep-th]) TWB, PJH, SR
- Forthcoming. . .

IIB superstrings on $AdS_5 \times S^5$



$$\frac{1}{\alpha'}\int d\tau d\sigma\,[\text{ non-linear } \sigma\text{-model }]$$

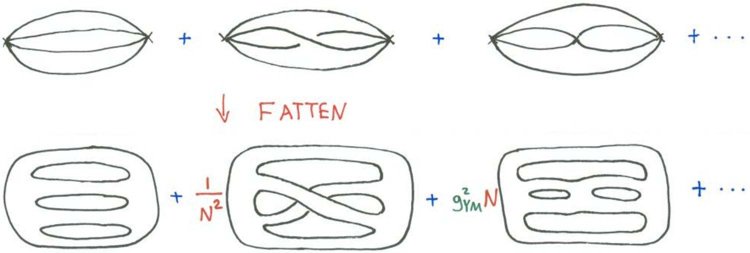
Perturbative expansion in the string coupling g_s

$\mathcal{N} = 4$ SUSY Yang-Mills: a Conformal Field Theory

$$\frac{N}{\lambda}\int d^4x\,\mathrm{tr}\Big[F_{\mu\nu}F^{\mu\nu} + D^\mu\phi_i D_\mu\phi_i - [\phi_i,\phi_j][\phi_i,\phi_j]$$

$$+\psi\sigma^\mu D_\mu\psi - \psi\phi\psi\Big] \;+\; \theta\int d^4x\,\mathrm{tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}]$$

Gauge group $U(N)$; fields in adjoint. Compute correlation functions of gauge-invariant local operators.



The AdS/CFT correspondence

$$\left\{ \begin{array}{c} \text{IIB superstrings on} \\ AdS_5 \times S^5 \end{array} \right\} = \left\{ \begin{array}{c} \mathcal{N} = 4 \text{ SUSY} \\ \text{Yang-Mills in 4d} \end{array} \right\}$$

$$\frac{R^2}{\alpha'} = \sqrt{\lambda}$$

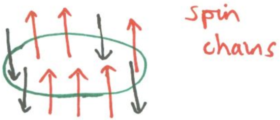
$$g_s = \frac{\lambda}{N}$$

bosonic symmetries $SO(2,4) \times SO(6)$ match

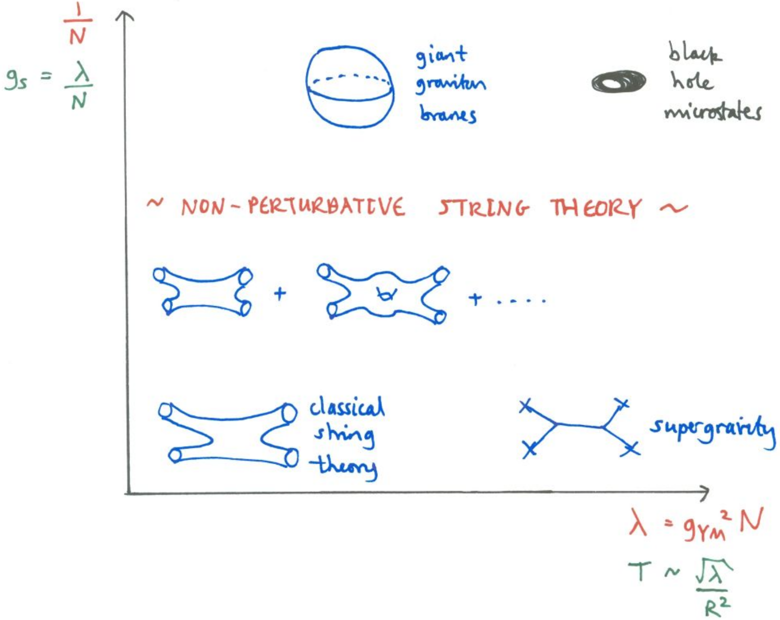
The Planar Limit

AdS/CFT has been successfully studied in the planar limit.
λ fixed, $N \to \infty \Rightarrow g_s \to 0$.

- Single-trace local operators in $\mathcal{N} = 4$ SYM.
- Classical string theory in bulk for strict $N \to \infty$ limit.
- Beautiful story of spinning strings, spin chains and integrability.



Parameter space



General Programme

We want to study AdS/CFT at finite N.

- Multi-trace and determinant-type operators in CFT.
- Must deal with Stringy Exclusion Principle.
- Correlation functions involve complicated combinatorics.
- Non-perturbative quantum gravity effects in the bulk, giant graviton branes, black holes.

Beasts of the field

In $\mathcal{N} = 4$ super Yang-Mills the fields are

$$X, Y, Z, X^\dagger, Y^\dagger, Z^\dagger; \lambda^A_\alpha, \bar{\lambda}^A_{\dot{\alpha}}; F_{\mu\nu} \quad \text{plus derivatives} \quad D_\mu$$

Each field is in the adjoint of the gauge group $U(N)$

$$(W_a)^i_j$$

$i, j = 1, 2 \ldots N$.
a runs over different fields.

To get gauge-invariant operators, usual route is to multiply these $N \times N$ matrices together and take traces

$$: \mathrm{tr}(XYX^\dagger)\, \mathrm{tr}(YZ)\, \mathrm{tr}(Y^\dagger) : \; = \; X^{i_1}_{i_2}\, Y^{i_2}_{i_3}\, X^{\dagger\, i_3}_{\;\; i_1} \;\; Y^{i_4}_{i_5}\, Z^{i_5}_{i_4} \;\; Y^{\dagger\, i_6}_{\;\; i_6}$$

Stringy Exclusion Principle

For an $N \times N$ matrix A, traces of powers bigger than N can always be written in terms of traces of powers $\leq N$

$$\mathrm{tr}(A^{N+p}) \; = \; \#\,\mathrm{tr}(A^N)\,\mathrm{tr}(A^p) \; + \; \#\,\mathrm{tr}(A^{N-1})\,\mathrm{tr}(A^p)\,\mathrm{tr}(A) + \cdots$$

For example, if $N = 2$ for the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\mathrm{tr}(A^3) = \frac{3}{2}\,\mathrm{tr}(A^2)\,\mathrm{tr}(A) - \frac{1}{2}\,\mathrm{tr}(A)\,\mathrm{tr}(A)\,\mathrm{tr}(A)$$

So working with traces is problematic...

Correlation functions

Wick contract with, e.g.,

$$\left\langle X^i_j(x) \;\; X^{\dagger k}{}_l(y) \right\rangle \;\; = \;\; \delta^i_l \delta^k_j \, \frac{1}{(x-y)^2}$$

Even at tree level this gives a complicated $\frac{1}{N}$ expansion

$$\left\langle \mathrm{tr}(XXXX)[x] \;\; \mathrm{tr}(X^\dagger X^\dagger X^\dagger X^\dagger)[y] \right\rangle = \left(4N^4 + 20N^2\right) \; \frac{1}{(x-y)^8}$$

$$\left\langle \mathrm{tr}(XXXX)[x] \;\; \mathrm{tr}(X^\dagger X^\dagger)\,\mathrm{tr}(X^\dagger X^\dagger)[y] \right\rangle = \left(16N^3 + 8N^1\right) \; \frac{1}{(x-y)^8}$$

Mixing between different trace structures is only suppressed when the length $n < N$. [For giant graviton $\Delta \sim N$, black hole $\Delta \sim N^2$.]

Outline of method

Solution: group theory.

Organise multi-trace operators of $\mathcal{N} = 4$ SYM at finite N into reps of the global symmetry group and reps of the local gauge group (which will control multi-trace structure à la Wilson loop).

Outline of method

Solution: group theory.

Organise multi-trace operators of $\mathcal{N}=4$ SYM at finite N into reps of the global symmetry group and reps of the local gauge group (which will control multi-trace structure à la Wilson loop).

1. Start with n fields with none of their indices contracted

$$(W_{a_1})^{i_1}_{j_1}\,(W_{a_2})^{i_2}_{j_2}\;\cdots\;(W_{a_n})^{i_n}_{j_n}$$

2. Build into reps of G and $U(N)$.
3. Enforce gauge invariance.

Technical slide 1/4: Example of $U(2)$

Take the fundamental representation V_F of $U(2)$

$$V_F = \begin{pmatrix} X \\ Y \end{pmatrix}$$

Technical slide 1/4: Example of $U(2)$

Take the fundamental representation V_F of $U(2)$

$$V_F = \begin{pmatrix} X \\ Y \end{pmatrix}$$

and then consider the simplest tensor product

$$V_F \otimes V_F$$

We can re-arrange into irreducible reps of $U(2)$

$$\begin{pmatrix} X \\ Y \end{pmatrix} \otimes \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \otimes X \\ X \otimes Y + Y \otimes X \\ Y \otimes Y \end{pmatrix} \oplus \begin{pmatrix} X \otimes Y - Y \otimes X \end{pmatrix}$$

$$\square \otimes \square = \square\!\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array}$$

Technical slide 2/4

We hit a problem with three copies of the fundamental

$$V_F^{\otimes 3} = \begin{pmatrix} X \otimes X \otimes X \\ \vdots \end{pmatrix} \oplus \begin{pmatrix} X \otimes X \otimes {\color{red}Y} - X \otimes {\color{red}Y} \otimes X \\ \vdots \end{pmatrix}$$
$$\oplus \begin{pmatrix} X \otimes X \otimes {\color{red}Y} - {\color{red}Y} \otimes X \otimes X \\ \vdots \end{pmatrix}$$

In terms of Young diagrams

$$\square \otimes \square \otimes \square = \ydiagram{3} \oplus {\color{red}2}\ \ydiagram{2,1}$$

How do we account for this multiplicity?

Technical slide 3/4: Schur-Weyl duality

For $V_F^{\otimes n}$, an n-tensor products of the fundamental of $U(K)$, there are two commuting group actions:

- $U(K)$: the action of $U(K)$ on its fundamental rep
- S_n: permutes the n different copies of V_F

Technical slide 3/4: Schur-Weyl duality

For $V_F^{\otimes n}$, an n-tensor products of the fundamental of $U(K)$, there are two commuting group actions:

- $U(K)$: the action of $U(K)$ on its fundamental rep
- S_n: permutes the n different copies of V_F

So organise $V_F^{\otimes n}$ in terms of representations of the two groups:

$$V_F^{\otimes n} \equiv \overbrace{\square \otimes \square \otimes \cdots \otimes \square}^{n} = \bigoplus_{\Lambda} V_{\Lambda}^{U(K)} \otimes V_{\Lambda}^{S_n}$$

where Λ runs over Young diagrams with n boxes and at most K rows.

[To answer question: $\dim V^{S_3}_{\tiny\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}} = 2$.]

Technical slide 4/4: Clebsch-Gordan coefficients

We can express this map from $V_F^{\otimes n}$ to reps of $U(K)$ and S_n using Clebsch-Gordan coefficients C.

$$C: \qquad V_F^{\otimes n} \qquad \rightarrow \quad V_\Lambda^{U(K)} \otimes V_\Lambda^{S_n}$$

$$C^{i_1 i_2 \dots i_n}_{\Lambda, M_\Lambda, m_\Lambda} \quad W_{i_1} \otimes W_{i_2} \otimes \cdots \otimes W_{i_n} \quad = \quad |\Lambda, M_\Lambda, m_\Lambda\rangle$$

- $i_k = \{1, 2, \dots K\}$ (for $U(2)$, $W_1 = X$, $W_2 = Y$)
- M_Λ labels $U(K)$ state in $V_\Lambda^{U(K)}$
- m_Λ labels S_n state in $V_\Lambda^{S_n}$
- Clebsch map is invertible

The solution: use C-G coefficients

Consider operators with n fields, a generic example being

$$(W_{a_1})^{i_1}_{j_1} \, (W_{a_2})^{i_2}_{j_2} \, \cdots \, (W_{a_n})^{i_n}_{j_n}$$

where $\{W_a\}$ are the fields of a subsector $G \subset PSU(2,2|4)$.

Combine indices into rep of $G \times S_n$ and two of $U(N) \times S_n$

$$|\Lambda(G), M_\Lambda, m_\Lambda\rangle \otimes |R(U(N)), M_R, m_R\rangle \otimes \left|\bar{S}(U(N)), M_S, m_S\right\rangle$$

$$= C^{a_1 \ldots a_n}_{\Lambda(G), M_\Lambda, m_\Lambda} \; C^{i_1 \ldots i_n}_{R(U(N)), M_R, m_R} \; C^{j_1 \ldots j_n}_{\bar{S}(U(N)), M_S, m_S} \; (W_{a_1})^{i_1}_{j_1} \cdots (W_{a_n})^{i_n}_{j_n}$$

- Enforce gauge invariance: pick singlet $\mathbf{1} \in R \otimes \bar{S}$ (implies $R = S$, sum over $M_R = M_S$)
- Impose overall S_n invariance

Simplest example: Half BPS Schur polynomials

For the $U(1)$ sector we only have one field: X. Thus we get

$$\sum_{M_R, m_R} C^{i_1 \ldots i_n}_{R(U(N)), M_R, m_R} \; C^{j_1 \ldots j_n}_{\bar{R}(U(N)), M_R, m_R} \; X^{i_1}_{j_1} \otimes \cdots \otimes X^{i_n}_{j_n}$$

$$= \frac{1}{n!} \sum_{\alpha \in S_n} \chi_R(\alpha) \; X^{i_1}_{i_{\alpha(1)}} X^{i_2}_{i_{\alpha(2)}} \cdots X^{i_n}_{i_{\alpha(n)}}$$

$$\equiv \chi_R(X)$$

- $U(N)$ rep R organises multi-trace structure (cf. Wilson loop).
- Encode finite N stringy exclusion principle, since reps of $U(N)$ have at most N rows.
- For $n \sim N$ map to giant gravitons, in general to LLM-type geometries.
- Can gain qualitative understanding of black hole microstates.

Diagonal Schur polynomials

Diagonal 2-point function

$$\left\langle \chi_R(X(x))\, \chi_S(X^\dagger(y)) \right\rangle = \delta_{RS}\ \mathrm{Dim}_N\ R\ \frac{1}{(x-y)^{2n}}$$

$\mathrm{Dim}_N\ R$ is the $U(N)$ dimension of R. It capture the N expansion, e.g.

$$Dim_N\ \ydiagram{3,2,1} = \frac{N^2(N+1)(N+2)(N-1)N(N-2)}{45}$$

The half-BPS sector is not renormalised, so this holds for all values of the coupling λ. This will not be true in general...

Subsectors

We can do this classification for the following sub-sectors $G \subset PSU(2,2|4)$ of the global superconformal symmetry group (and product groups $G_1 \times G_2$) :

$$
\begin{aligned}
\text{half BPS } U(1): \quad \{W_m\} &= \{X\} \\
U(3): \quad \{W_m\} &= \{X, Y, Z\} \\
U(3|2): \quad \{W_m\} &= \{X, Y, Z; \psi_1, \psi_2\} \\
O(2): \quad \{W_m\} &= \{X, X^\dagger\} \\
SL(2): \quad \{W_m\} &= \{X, \partial X, \partial^2 X, \partial^3 X, \dots\} \\
SO(2,4): \quad \{W_m\} &= \{X, \partial_\mu X, \partial_\mu \partial_\nu X, \dots\}
\end{aligned}
$$

Operator for general G

$$\mathcal{O}\big[\overbrace{\Lambda(G),\, M_\Lambda}^{\text{rep and state of } G},\, R(U(N)),\, \tau\big]$$

R of $U(N)$ gives multi-trace structure (multiplicity)

- Complete basis on space of multi-trace operators at finite N built out of fundamental fields of G.
- Free 2-point function totally diagonal on all labels, proportional to $\mathrm{Dim}_N\ R$.
- Operators given in detail for $G = U(3), SL(2), O(2), SO(2,4)$, prescription given for $SO(6)$.
- For $SL(2)$, in regime of large quantum numbers, spectrum of our basis matches excitations of (non-BPS) giant gravitons.

One loop

At one loop this basis is no longer diagonal. Operators mix and we must rediagonalise. Multiplets also re-organise in a highly non-trivial way. Take for example the $U(2)$ sector, $\Lambda = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}$.

The $\frac{1}{4}$-BPS operators, which are protected, are in 1-to-1 correspondence with the chiral ring and receive $\frac{1}{N}$ corrections, e.g.

$$\mathrm{tr}(XX)\,\mathrm{tr}(YY) - \mathrm{tr}(XY)\,\mathrm{tr}(XY) - \frac{1}{N} tr([X,Y][X,Y])$$

Some operators are no longer protected and join long multiplets

$$tr([X,Y][X,Y]) \qquad \Delta = 4 + \frac{3\lambda}{4\pi^2} + \mathcal{O}(\lambda^2)$$

(This becomes a descendant of the Konishi.)

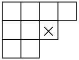
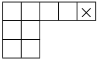
Constrained mixing at one loop

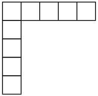
Analyse mixing with one-loop dilatation operator, e.g. $U(2)$ sector

$$: \mathrm{tr}([X, Y][\tilde{X}, \tilde{Y}]) :$$

$\tilde{X} \sim \frac{\partial}{\partial X}$. This gives matrix of anomalous dimensions.

The $U(N)$ representations, controlling multi-trace structure, then only mix if related by repositioning a single box.

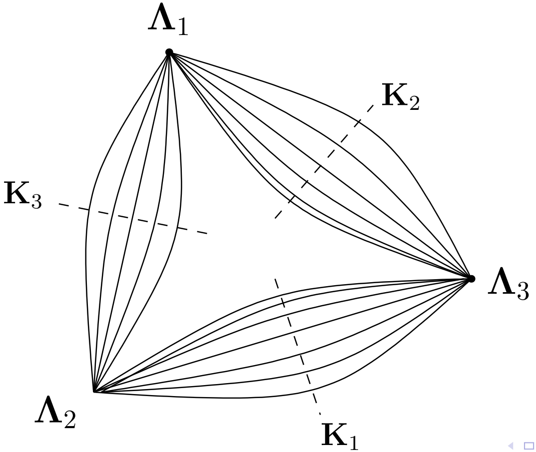
$R =$  mixes with $S =$ 

but not with $T =$ 

Free three-point function

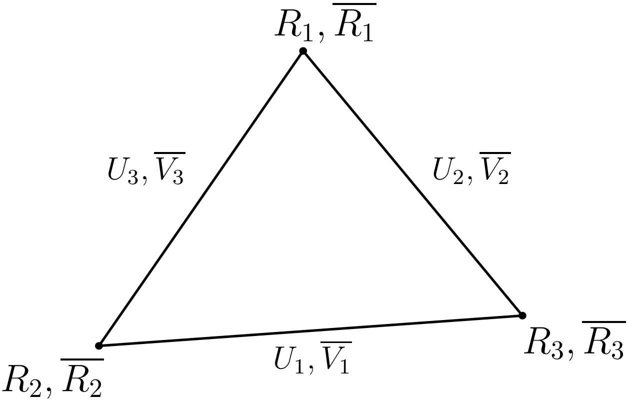
We can also use this formalism to work out the free non-extremal three-point function

$$\left\langle \mathcal{O}[\boldsymbol{\Lambda}_1, R_1](x_1)\ \ \mathcal{O}[\boldsymbol{\Lambda}_2, R_2](x_2)\ \ \mathcal{O}[\boldsymbol{\Lambda}_3, R_3](x_3) \right\rangle$$



Three-point gauge spin network

On the legs between the operators the gauge group representations need not form a singlet. The three-point function becomes a $G \times U(N)$ spin network.



Conclusions

- For sectors G of $\mathcal{N} = 4$ global symmetry group multi-trace operators organised into a complete basis that transforms in irreps of G, traces organised by $U(N)$ irreps.
- This basis diagonalises the free two-point function, including all finite N corrections.
- One-loop mixing nicely constrained.
- Higher-point functions in free theory form $G \times U(N)$ spin networks. (Free theory $\sim$ finite N tensionless 'string'.)
- Focus in future:
  - Extend to full $PSU(2,2|4)$ symmetry group.
  - Diagonalise spectrum at 1-loop.
  - Sixteenth-BPS states: how do they furnish black hole entropy?
  - Understand information loss.
  - What is string theory?