Cut-and-join operators and $\mathcal{N} = 4$ SYM

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General Programme

- Study $\frac{1}{N}$ corrections to $\mathcal{N} = 4, d = 4$ super Yang-Mills with guage group $U(N)$.
- Multi-trace operators with $\Delta_0 \equiv n < N^{\frac{1}{2}}$. Organise into:
  - Representations of the global symmetry group;
  - Operators with fixed trace structure, e.g. single/double trace.
- Focus on theory at tree level and one loop.
  - Messy mixing problem;
  - Want to find operators with well-defined conformal dimensions;
  - Is there a string dual to the free gauge theory?

Two different attitudes

Two different attitudes to $\frac{1}{N}$ corrections, depending on coupling.

- For free theory, $\lambda = 0$, treat $\frac{1}{N}$ as a string coupling ordering the non-planar expansion of correlation functions. Multi-trace operators identified with multi-string states.

- For $\lambda > 0$ the correct string expansion is in $g_s = \frac{\lambda}{N}$. Treat $\frac{1}{N}$ corrections as a modification to the gauge theory/string theory state identification.

Review of half-BPS sector

Based on Vaman and Verline 0209215; Corley, Jevicki and Ramgoolam 0111222.

Trace structures of operators map to conjugacy classes of S_n.
E.g. for $\alpha = (123)(45)(6) \in S_6$

$$\begin{aligned} \mathrm{tr}(X^3)\,\mathrm{tr}(X^2)\,\mathrm{tr}(X) &= X^{i_1}_{i_2} X^{i_2}_{i_3} X^{i_3}_{i_1} X^{i_4}_{i_5} X^{i_5}_{i_4} X^{i_6}_{i_6} \\ &= X^{i_1}_{i_{\alpha(1)}} X^{i_2}_{i_{\alpha(2)}} X^{i_3}_{i_{\alpha(3)}} X^{i_4}_{i_{\alpha(4)}} X^{i_5}_{i_{\alpha(5)}} X^{i_6}_{i_{\alpha(6)}} \end{aligned}$$

Conjugacy classes labelled by partitions of n, e.g. $[3,2,1]$ here.

Two-point function given by cut-and-join operators

$$\Big\langle \mathrm{tr}(\alpha' X^{\dagger n})\ \mathrm{tr}(\alpha X^n) \Big\rangle_{\text{non-planar}} = N^n \langle \alpha' \big| \Omega_n \left| \alpha \right\rangle$$

(We're dropping the spacetime dependence here and onwards.)

Cut-and-join operators

Basic cut-and-join operator is a sum over the transpositions in S_n

$$\Sigma_{[2]} = \sum_{i<j} (ij)$$

It cuts a single trace/cycle $[n] = (123 \cdots n)$ into two

$$\Sigma_{[2]} \, |n\rangle \sim |n_1, n_2\rangle$$

It both joins a double trace and cuts it into three

$$\Sigma_{[2]} \, |n_1, n_2\rangle \sim |n\rangle + |n_1, n_2, n_3\rangle$$

Tree-level mixing given by

$$\begin{aligned} \Omega_n &= \sum_{\sigma \in S_n} \frac{1}{N^{T(\sigma)}} \, \sigma \\ &= 1 + \frac{1}{N} \Sigma_{[2]} + \frac{1}{N^2} \left(\Sigma_{[3]} + \Sigma_{[2,2]} \right) + \mathcal{O}\left(\frac{1}{N^3} \right) \end{aligned}$$

Inner product and full non-planar correlation function

The inner product is given by the leading *planar* two-point function

$$\langle \alpha' | \alpha \rangle \sim \delta_{\alpha' \in [\alpha]}$$

The leading term of the (extremal) three-point function

$$\langle n_1, n_2 | \left(\frac{1}{N} \Sigma_{[2]} \right) | n \rangle = \frac{n n_1 n_2}{N}$$

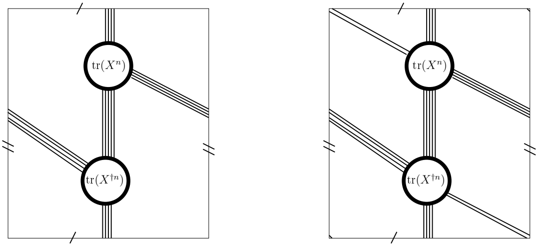
The first correction to the single-trace 2-p't f'n from the torus

$$\langle n | \left(\frac{1}{N^2} \left[\Sigma_{[3]} + \Sigma_{[2,2]} \right] \right) | n \rangle = \frac{n}{N^2} \left[\binom{n}{3} + \binom{n}{4} \right]$$

What do these numbers mean in a putative worldsheet theory?

Bunching of homotopic propagators

The $\Sigma_{[3]}$ term gives propagators on the torus bunched into 3 groups; $\Sigma_{[2,2]}$ gives propagators bunched into 4 groups.



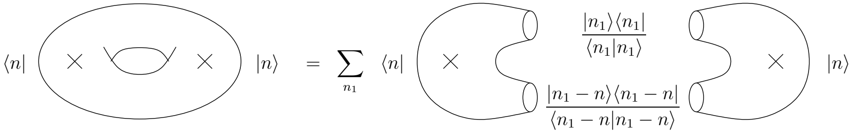
In Gopakumar's model, each Σ_C gives a different skeleton graph of homotopically-bunched propagators for the relevant genus g.

Suggestively, these are Hurwitz numbers counting n-branched covers of $\mathbb{C}P^1$ by surfaces of genus g with three branch points, two labelled by the operators and the third by the cut-and-join Σ_C.

Two-dimensional factorisation of correlation functions

Another feature is that for large n the higher genus correlation functions factorise into planar 3-point functions, e.g. for torus

$$\frac{1}{N^2}\left(\Sigma_{[3]}+\Sigma_{[2,2]}\right) \to \frac{1}{2}\left(\frac{1}{N}\Sigma_{[2]}\right)^2$$



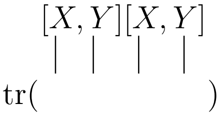
This is the result of the exponentiation of the tree-level mixer

$$\Omega_n = \exp\left(\frac{1}{N}\Sigma_{[2]} - \frac{1}{2N^2}\left[\binom{n}{2}+\Sigma_{[3]}\right] + \mathcal{O}\left(\frac{1}{N^3}\right)\right)$$
$$\to \exp\left(\frac{1}{N}\Sigma_{[2]}\right)$$

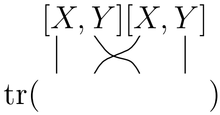
NB: additional terms subleading in $\frac{n^2}{N}$.

Multiple fields: a few simple examples I

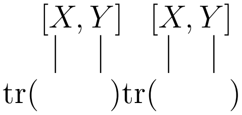
Tracing the same field content for $U(2) \subset SU(4)_R$ rep $\Lambda = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}$ we sometimes have to 'twist' the trace to get a non-vanishing operator



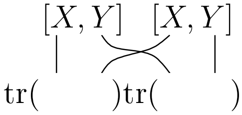
$$= \mathrm{tr}([X,Y][X,Y])$$



$$= 0$$



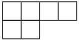
$$= 0$$



$$= \mathrm{tr}(\Phi^r\Phi^s)\mathrm{tr}(\Phi_r\Phi_s)$$

where $\Phi^p\Phi_p = \epsilon^{pq}\Phi_p\Phi_q = [X,Y]$.

Multiple fields: a few simple examples II

Things also get complicated when for a given representation and trace structure there is more than one operator, e.g. for the $U(2)$ rep  $\sim [X,Y][X,Y]\,XX$ with trace structure [4, 2]

$$\mathrm{tr}([X,Y][X,Y])\,\mathrm{tr}(XX)$$

$$\mathrm{tr}(XX\Phi^r\Phi^s)\,\mathrm{tr}(\Phi_r\Phi_s)$$

(remembering that $\Phi^p\Phi_p = \epsilon^{pq}\Phi_p\Phi_q = [X,Y]$).

Solution for multiple fields

For $U(2)$ sector organise n copies of fields $\{X, Y\}$ into reps

$$V_{\mathbf{2}}^{\otimes n} = \bigoplus_{|\Lambda|=n} V_\Lambda^{U(2)} \otimes V_\Lambda^{S_n}$$

Can then write all multitrace operators as

$$|\Lambda, M; \alpha, \gamma\rangle \equiv \frac{1}{n!} \sum_{\sigma \in S_n} S_{a\gamma}^{\Lambda,\alpha}\ B_{b\beta}^{\Lambda,\vec{\mu}}\ D_{ab}^{\Lambda}(\sigma)\ \mathrm{tr}(\sigma^{-1}\alpha\sigma\ \overbrace{X\cdots X}^{\mu_1}\ \overbrace{Y\cdots Y}^{\mu_2})$$

- Λ tells us the rep. of $U(2)$ (a two-row n-box Young diagram)
- M tells us the state within that rep.
- α is a partition of n giving the trace structure
- γ labels the multiplicity for this Λ and α; no. of values is

$$\frac{1}{|\mathrm{Sym}(\alpha)|} \sum_{\rho \in \mathrm{Sym}(\alpha)} \chi_\Lambda(\rho)$$

Example operators

$$\left|\Lambda = \yng(2,2), M = HWS; \alpha = [4], \gamma = 1\right\rangle = \mathrm{tr}([X, Y][X, Y])$$

$$\left|\Lambda = \yng(2,2), M = HWS; \alpha = [2,2], \gamma = 1\right\rangle = \mathrm{tr}(\Phi^r \Phi^s)\,\mathrm{tr}(\Phi_r \Phi_s)$$

$$\left|\yng(4,2), HWS; [4,2], 1\right\rangle = \mathrm{tr}([X, Y][X, Y])\,\mathrm{tr}(XX)$$

$$\left|\yng(4,2), HWS; [4,2], 2\right\rangle = \mathrm{tr}(XX\Phi^r \Phi^s)\,\mathrm{tr}(\Phi_r \Phi_s) + \frac{1}{6}\,\mathrm{tr}([X, Y][X, Y])\,\mathrm{tr}(XX)$$

Inner product and non-planar 2-point function

The inner product (i.e. *planar* two-point function) is diagonal

$$\langle \Lambda', M'; \alpha', \gamma' | \Lambda, M; \alpha, \gamma \rangle \propto \delta^{\Lambda\Lambda'} \delta^{MM'} \delta^{\alpha\alpha'} \delta^{\gamma\gamma'}$$

As for the half-BPS sector, the cut-and-join operators give the full non-planar free two-point function

$$\left\langle \mathcal{O}^{\dagger}[\Lambda', M'; \alpha', \gamma'] \;\; \mathcal{O}[\Lambda, M; \alpha, \gamma] \right\rangle_{\text{non-planar}} = \delta^{\Lambda\Lambda'} \delta^{MM'} \; N^n \; \langle \Lambda, M; \alpha', \gamma' | \; \Omega_n \; | \Lambda, M; \alpha, \gamma \rangle$$

From $U(2)$ to $PSU(2,2|4)$

This works automatically for $U(2) \to U(K_1|K_2)$. To extend these results for the free theory to the other fields of $\mathcal{N} = 4$ SYM treat the infinite-dimensional singleton rep. of $PSU(2,2|4)$ as the fundamental of $U(\infty|\infty)$. (The Λ are now unrestricted S_n reps, also known as the higher spin YT-pletons.)

However as soon as we turn on the coupling the $PSU(2,2|4)$ group structure asserts itself. Each rep Λ breaks down into an infinite number of $PSU(2,2|4)$ reps. This decomposition is tricky and not known in general. Using the technology of Schur-Weyl duality we *can* do this for e.g. $SO(6)$ and $SO(2,4)$.

One-loop

Analyse mixing with one-loop dilatation operator, e.g. $U(2)$ sector

$$:\mathrm{tr}([X,Y][\tfrac{\partial}{\partial X},\tfrac{\partial}{\partial Y}]):$$

Operators with anomalous dimensions have commutators $[X,Y]$ within a trace. Label them $|\Lambda, M; \alpha^a, \gamma^a\rangle$, e.g.

$$\left|\yng(2,2), HWS; [4]^a, 1^a\right\rangle = \mathrm{tr}([X,Y][X,Y])$$

$$\left|\yng(4,2), HWS; [4,2]^a, 1^a\right\rangle = \mathrm{tr}([X,Y][X,Y])\,\mathrm{tr}(XX)$$

How do we find the quarter-BPS operators?

On general grounds the protected BPS operators must be orthogonal to those operators with anomalous dimensions in the full non-planar two-point function. So choose α^q, γ^q such that

$$\langle \Lambda, M; \alpha^a, \gamma^a | \Lambda, M; \alpha^q, \gamma^q \rangle = 0 \qquad \forall a, q$$

The $\frac{1}{4}$-BPS ops. are defined with the inverse of the tree-level mixer

$$\boxed{\tfrac{1}{4}\text{-BPS } = \Omega_n^{-1} \; |\Lambda, M; \alpha^q, \gamma^q\rangle}$$

for $\quad \Omega_n^{-1} = 1 - \frac{1}{N}\Sigma_{[2]} + \frac{1}{N^2}\left[\frac{n(n-1)}{2} + 2\Sigma_{[3]} + \Sigma_{[2,2]}\right] + \mathcal{O}\left(\frac{1}{N^3}\right)$

Quarter-BPS examples

$$\Omega_n^{-1}\left|\yng(2,2), HWS; [2,2]^q, 1^q\right\rangle = \mathrm{tr}(\Phi^r\Phi^s)\,\mathrm{tr}(\Phi_r\Phi_s) + \frac{2}{N}\,\mathrm{tr}([X,Y][X,Y]) - \frac{2}{N^2}\,\mathrm{tr}(\Phi^r\Phi^s)\,\mathrm{tr}(\Phi_r)\,\mathrm{tr}(\Phi_s)$$

$$\begin{aligned}
&\Omega_n^{-1}\left|\yng(4,2), HWS; [4,2]^q, 2^q\right\rangle \\
&= \mathrm{tr}(XX\Phi^r\Phi^s)\,\mathrm{tr}(\Phi_r\Phi_s) + \frac{1}{6}\,\mathrm{tr}([X,Y][X,Y])\,\mathrm{tr}(XX) \\
&\quad + \frac{8}{3N}\,\mathrm{tr}(\Phi^r\Phi_r\Phi^s\Phi_s XX) - \frac{16}{3N}\,\mathrm{tr}(\Phi^r\Phi^s\Phi_r\Phi_s XX) \\
&\quad - \frac{4}{3N}\,\mathrm{tr}(\Phi^r\Phi^s)\,\mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(XX) \\
&\quad - \frac{1}{N}\,\mathrm{tr}(\Phi^r\Phi^s XX)\,\mathrm{tr}(\Phi_r)\,\mathrm{tr}(\Phi_s) - \frac{1}{6N}\,\mathrm{tr}(\Phi^r\Phi_r\Phi^s\Phi_s)\,\mathrm{tr}(X)\,\mathrm{tr}(X) \\
&\quad - \frac{4}{N}\,\mathrm{tr}(\Phi^r\Phi^s X)\,\mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(X) + \frac{2}{N}\,\mathrm{tr}(\Phi^r\Phi^s X)\,\mathrm{tr}(\Phi_r X)\,\mathrm{tr}(\Phi_s) + \mathcal{O}\left(\frac{1}{N^2}\right)
\end{aligned}$$

Conclusions

- Full non-planar free theory has a universal structure given by cut-and-join operators, with many stringy features.
    - Can we turn this into a concrete description of the dual string?
- Some features also appear in the weak coupling regime, at least in identifying the quarter-BPS operators.
    - Does any of this apply to ops with anomalous dimensions?