$PSL(n|n)$ Lie superalgebra (co)homology

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1 Lie superalgebra relations

1.1 Quadratic twistor free field representation

For $psl(n|n)$ non-affine singleton have bosonic and fermionic oscillators for $\alpha, \beta, i, j = 1, \dots n$

$$\begin{aligned} [a^\alpha, a^\dagger_\beta] &= \delta^\alpha_\beta \\ \{c^i, c^\dagger_j\} &= \delta^i_j \end{aligned} \tag{1}$$

Introduce block-diagonal bosonic generators

$$\begin{aligned} L^\alpha_\beta &= a^\dagger_\beta a^\alpha - \tfrac{1}{n}\delta^\alpha_\beta a^\dagger_\gamma a^\gamma \\ R^i_j &= c^\dagger_j c^i - \tfrac{1}{n}\delta^i_j c^\dagger_k c^k \\ C &= a^\dagger_\alpha a^\alpha + c^\dagger_i c^i \\ B &= n + a^\dagger_\alpha a^\alpha - c^\dagger_i c^i \end{aligned} \tag{2}$$

C commutes with everything, B has non-trivial commutators with the fermionic generators.

The off-diagonal fermionic generators are

$$\begin{aligned} Q^i_\alpha &= a^\dagger_\alpha c^i \\ \bar{Q}^\alpha_i &= c^\dagger_i a^\alpha \end{aligned} \tag{3}$$

Take the Cartans to be

$$(L^1_1, \dots L^{n-1}_{n-1}, R^1_1, \dots R^{n-1}_{n-1}) \tag{4}$$

Sometimes we will group everything together into a matrix M^a_b with $a, b = 1, \dots 2n$ with

$$M^a_b = M^{\alpha,i}_{\beta,j} = \begin{pmatrix} L^\alpha_\beta & \bar{Q}^\alpha_j \\ Q^i_\beta & R^i_j \end{pmatrix} \tag{5}$$

The non-trivial commutation relations are

$$
\begin{aligned}
&[L^\alpha_\beta, L^\gamma_\delta] = \delta^\alpha_\delta L^\gamma_\beta - \delta^\gamma_\beta L^\alpha_\delta \\
&[R^i_j, R^k_l] = \delta^i_l R^k_j - \delta^k_j R^i_l \\
&[L^\alpha_\beta, Q^i_\gamma] = \delta^\alpha_\gamma Q^i_\beta - \tfrac{1}{n}\delta^\alpha_\beta Q^i_\gamma \\
&[L^\alpha_\beta, \bar{Q}^\gamma_i] = -\delta^\gamma_\beta \bar{Q}^\alpha_i + \tfrac{1}{n}\delta^\alpha_\beta \bar{Q}^\gamma_i \\
&[R^i_j, Q^k_\alpha] = -\delta^k_j Q^i_\alpha + \tfrac{1}{n}\delta^i_j Q^k_\alpha \\
&[R^i_j, \bar{Q}^\alpha_k] = \delta^i_k \bar{Q}^\alpha_j - \tfrac{1}{n}\delta^i_j \bar{Q}^\alpha_k \\
&\{Q^i_\alpha, \bar{Q}^\beta_j\} = \delta^i_j L^\beta_\alpha + \delta^\beta_\alpha R^i_j + \tfrac{1}{n}\delta^i_j \delta^\beta_\alpha C \\
&[B, Q^i_\alpha] = 2Q^i_\alpha \\
&[B, \bar{Q}^\alpha_i] = -2\bar{Q}^\alpha_i
\end{aligned}
\tag{6}
$$

2 BRST operator

We have ghost pairs (C^a_b, B^c_d) with commutator

$$[B^a_b, C^c_d] = \delta^a_d \delta^c_b \tag{7}$$

Sometimes we will use (γ, β) for the bosonic ghosts.

The BRST operator is then

$$
\begin{aligned}
Q =& C^\beta_\alpha L^\alpha_\beta - C^n_n L^n_n + C^j_i R^i_j - C^{2n}_{2n} R^n_n + \bar{\gamma}^\alpha_i Q^i_\alpha + k\gamma^i_\alpha \bar{Q}^\alpha_i \\
&- C^\beta_\alpha C^\alpha_\gamma B^\gamma_\beta + C^n_\alpha C^\alpha_n \textstyle\sum_{a=1}^{n-1} B^a_a \\
&- C^j_i C^i_k B^k_j + C^{2n}_i C^i_{2n} \textstyle\sum_{a=n+1}^{2n-1} B^a_a \\
&- C^\alpha_\beta \bar{\gamma}^\beta_i \beta^i_\alpha + \tfrac{1}{n} C^\alpha_\alpha \bar{\gamma}^\beta_i \beta^i_\beta \\
&+ k C^\alpha_\beta \gamma^i_\alpha \bar{\beta}^\beta_i - \tfrac{k}{n} C^\alpha_\alpha \gamma^i_\beta \bar{\beta}^\beta_i \\
&+ C^i_j \bar{\gamma}^\beta_i \beta^j_\beta - \tfrac{1}{n} C^i_i \bar{\gamma}^\alpha_j \beta^j_\alpha \\
&- k C^i_j \gamma^j_\alpha \bar{\beta}^\alpha_i + \tfrac{k}{n} C^i_i \gamma^j_\alpha \bar{\beta}^\alpha_j \\
&- k\bar{\gamma}^\alpha_i \gamma^i_\beta B^\beta_\alpha + k\bar{\gamma}^n_i \gamma^i_n \textstyle\sum_{a=1}^{n-1} B^a_a \\
&- k\bar{\gamma}^\alpha_i \gamma^j_\alpha B^i_j + k\bar{\gamma}^\alpha_n \gamma^n_\alpha \textstyle\sum_{a=n+1}^{2n-1} B^a_a
\end{aligned}
\tag{8}
$$

Ignore Einstein summation terms that include C^n_n and C^{2n}_{2n}.

The sums from 1 to $n-1$ come from commutations that give L^n_n or R^n_n that we want to replace.

k should be -1 to get a proper metric, but it's easier just to set it to be 1.

3 BRST cohomology for trivial representation

In this case we have states

$$C^{A_1} \cdots C^{A_p} |0\rangle \tag{9}$$

The BRST operator we need is

$$Q = f^{AB}_C : C^{\hat{A}} C^{\hat{B}} B_{\hat{C}} : \tag{10}$$

and it acts

$$QC^{A_1} \cdots C^{A_p} |0\rangle = f^{AB}_C C^{\hat{A}} C^{\hat{B}} [B_{\hat{C}}, C^{A_1} \cdots C^{A_p}] |0\rangle \tag{11}$$

ghost number	0	1	2	3	4	5	6	$2n$	$2n+1$
total	1	0	3	1	5	3	7	$2n+1$	$2n-1$

Table 1: Number of states in cohomology for trivial representation of $psl(2|2)$.

ghost number	0	1	2	3	4	5
total	1	0	1	3	1	4

Table 2: Number of states in cohomology for trivial representation of $psl(3|3)$.

ghost number	0	1	2	3	4
total	1	0	1	1	3

Table 3: Number of states in cohomology for trivial representation of $psl(4|4)$.

4 Singleton representation for n even

n must be even for the singleton representation.

Vacuum $|1\rangle$ defined by

$$\begin{aligned} a^\dagger_\alpha |1\rangle = c^\dagger_i |1\rangle = 0 \quad &\text{for } \alpha, i = 1, \dots \tfrac{n}{2} \\ a^\alpha |1\rangle = c^i |1\rangle = 0 \quad &\text{for } \alpha, i = \tfrac{n}{2}+1, \dots n \end{aligned} \tag{12}$$

The vacuum $|1\rangle$ is killed by the central element C from (2), $C|1\rangle = 0$. With the slightly awkward definition of B in (2) we also have $B|1\rangle = 0$.

It is annihilated by three-quarters of the fermionic generators:

$$\begin{aligned} Q^i_\alpha |1\rangle = 0 \quad &\text{for } i = \tfrac{n}{2}+1, \dots n \text{ and } \alpha = 1, \dots \tfrac{n}{2} \\ \bar{Q}^\alpha_i |1\rangle = 0 \quad &\text{for } \alpha = \tfrac{n}{2}+1, \dots n \text{ and } i = 1, \dots \tfrac{n}{2} \end{aligned} \tag{13}$$

These include all of the lowering operators and half of the raising operators, making it half-BPS.

Acting with the remaining generators

$$\begin{aligned} Q^i_\alpha \quad &\text{for } i = 1, \dots \tfrac{n}{2} \text{ and } \alpha = \tfrac{n}{2}+1, \dots n \\ \bar{Q}^\alpha_i \quad &\text{for } \alpha = 1, \dots \tfrac{n}{2} \text{ and } i = \tfrac{n}{2}+1, \dots n \end{aligned} \tag{14}$$

gives us the $psl(n|n)$ singleton representation, which coincides with all the oscillator states we can build on this vacuum with $C = 0$.

The Cartans on the vacuum are

$$(L^1_1, \dots L^{\frac{n}{2}}_{\frac{n}{2}}, L^{\frac{n}{2}+1}_{\frac{n}{2}+1}, \dots L^{n-1}_{n-1}, R^1_1, \dots R^{\frac{n}{2}}_{\frac{n}{2}}, R^{\frac{n}{2}+1}_{\frac{n}{2}+1}, \dots R^{n-1}_{n-1}) |1\rangle = (-\tfrac{1}{2}, \dots -\tfrac{1}{2}, \tfrac{1}{2} \dots \tfrac{1}{2}, \tfrac{1}{2}, \dots \tfrac{1}{2}, -\tfrac{1}{2}, \dots -\tfrac{1}{2}) |1\rangle \tag{15}$$

The commutators of the Cartans with the oscillators are $[L^\beta_\beta, a^\dagger_\alpha] = \delta^\beta_\alpha a^\dagger_\beta - \frac{1}{n} a^\dagger_\alpha$

$$\begin{aligned} [L^\beta_\beta, a^\dagger_\alpha] &= \delta^\beta_\alpha a^\dagger_\beta - \tfrac{1}{n} a^\dagger_\alpha \\ [L^\beta_\beta, a^\alpha] &= -\delta^\alpha_\beta a^\beta + \tfrac{1}{n} a^\alpha \\ [R^j_j, c^\dagger_i] &= \delta^j_i c^\dagger_j - \tfrac{1}{n} c^\dagger_i \\ [R^j_j, c^i] &= -\delta^i_j c^j + \tfrac{1}{n} c^i \end{aligned} \tag{16}$$

To build the states of the singleton, construct states with numbers of the oscillators $n^a_\alpha \in \{0, \dots\}$ for $\alpha \in \{1, \dots n\}$ and $n^c_i \in \{0, 1\}$ for $i \in \{1, \dots n\}$

$$\prod_{\alpha=1}^{\frac{n}{2}} (a^\alpha)^{n^a_\alpha} \prod_{\alpha=\frac{n}{2}+1}^{n} (a^\dagger_\alpha)^{n^a_\alpha} \prod_{i=1}^{\frac{n}{2}} (c^i)^{n^c_i} \prod_{i=\frac{n}{2}+1}^{n} (c^\dagger_i)^{n^c_i} |1\rangle \tag{17}$$

and then impose the $C = 0$ constraint.

Given the $2n-2$ Cartans and the C and B of such a state, the number of oscillators is then

$$\begin{aligned}
n^a_\alpha &= -\tfrac{1}{2} - \tfrac{1}{2n}B - \tfrac{1}{2n}C - L^\alpha_\alpha && \text{for } \alpha \in \{1, \dots \tfrac{n}{2}\} \\
n^a_\alpha &= -\tfrac{1}{2} + \tfrac{1}{2n}B + \tfrac{1}{2n}C + L^\alpha_\alpha && \text{for } \alpha \in \{\tfrac{n}{2}+1, \dots n-1\} \\
n^a_n &= -\tfrac{1}{2} + \tfrac{1}{2n}B + \tfrac{1}{2n}C - \textstyle\sum_{\beta=1}^{n-1} L^\beta_\beta \\
n^c_i &= \tfrac{1}{2} + \tfrac{1}{2n}B - \tfrac{1}{2n}C - R^i_i && \text{for } i \in \{1, \dots \tfrac{n}{2}\} \\
n^c_i &= \tfrac{1}{2} - \tfrac{1}{2n}B + \tfrac{1}{2n}C + R^i_i && \text{for } i \in \{\tfrac{n}{2}+1, \dots n-1\} \\
n^c_n &= \tfrac{1}{2} - \tfrac{1}{2n}B + \tfrac{1}{2n}C - \textstyle\sum_{j=1}^{n-1} R^j_j
\end{aligned} \tag{18}$$

Given that there are only so many Q's and $\bar{Q}$'s we can use to build up the states from (14) and the commutation relations (6), we find that for the singleton states $|B| \in \{0, 2, 4, \dots n\}$. Note that with these conventions for $psl(4|4)$, $|B| = 4$ corresponds to the field strength, $|B| = 2$ corresponds to the fermions and $B = 0$ corresponds to the scalars.

$\mathcal{N} = 4$ fields	oscillator states	Cartans	B
$F_{(1,2)(1,2)}$	$a^\dagger_{(3,4)} a^\dagger_{(3,4)} c^0 c^1 \|1\rangle$	$[-1, -1, p, 0, 0, 0]$	4
$\lambda^i_{(3,4)}$	$\{a^\dagger_{(3,4)} c^{(1,2)} \|1\rangle, a^\dagger_{(3,4)} c^1 c^2 c^\dagger_{(3,4)} \|1\rangle\}$	$\cdots$	2
Z	$\|1\rangle$	$[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$	0
$\{X, Y, X^\dagger, Y^\dagger\}$	$c^{(0,1)} c^\dagger_{(3,4)} \|1\rangle$	$[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}]$	0
$Z^\dagger$	$c^0 c^1 c^\dagger_3 c^\dagger_4 \|1\rangle$	$[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]$	0
$\bar{\lambda}^{(1,2)}_i$	$\{a^{(1,2)} c^\dagger_{(3,4)} \|1\rangle, a^{(1,2)} c^{(1,2)} c^\dagger_3 c^\dagger_4 \|1\rangle\}$	$\cdots$	-2
$\bar{F}^{(1,2)(1,2)}$	$a^{(1,2)} a^{(1,2)} c^\dagger_3 c^\dagger_4 \|1\rangle$	$[-p, -2+p, 1, 0, 0, 0]$	-4
$\partial^{(1,2)}_{(3,4)}$	$a^{(1,2)} a^\dagger_{(3,4)}$	$\cdots$	0

Table 4: Map to states. $p \in \{0, 1, 2\}$.

Note that F_{12} and $\bar{F}^{12}$ have the same $psl(4|4)$ Cartans and can only be distinguished by their B charge.

5 BRST cohomology for the singleton

Because of the large growth of states the results here are a bit limited.

ghost number	0	1	2	3	4	5
total	0	2	0	4	2	6

Table 5: Number of states in cohomology for singleton representation of $psl(2|2)$.

ghost number	0	1	2	3
total	0	0	2	0

Table 6: Number of states in cohomology for singleton representation of $psl(4|4)$.