Reducing partition algebras

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Contents

1 The problem

Consider a vector of n numbers $\vec{x} = \{x_1, x_2, \dots x_n\}$. This has a natural action of the symmetric group: for $\sigma \in S_n$

$$x_i \mapsto x_{\sigma(i)} \tag{1}$$

We will call this n-dimensional representation the *natural* representation of S_n, written $V_{\text{nat}}^{S_n}$. In the literature it is also known as the permutation representation. It happens to be reducible $V_{\text{nat}}^{S_n} = V_{[n]}^{S_n} \oplus V_{[n-1,1]}^{S_n}$ but we won't use that property here.

k-fold tensor products of the natural representation can be decomposed into representations of S_n and its Schur-Weyl dual, the partition algebra $P_k(n)$. Since there is also an action of $\mathbb{C}S_k \subset P_k(n)$ on this space, we can further decompose each representation $V_\lambda^{P_k(n)}$ of the partition algebra into representations $V_\kappa^{S_k}$ of S_k. These generically come with a (possibly zero) multiplicity, which we will label $V_{\lambda,\kappa}^{C(S_n,S_k)}$, named such because it is somehow the rep of the commutant of $S_n \times S_k$ in the space of endomorphisms of $(V_{\mathrm{nat}}^{S_n})^{\otimes k}$.

$$
\begin{aligned}
\left(V_{\mathrm{nat}}^{S_n}\right)^{\otimes k} &= \bigoplus_{\lambda \text{ of } S_n, P_k(n)} V_\lambda^{S_n} \otimes V_\lambda^{P_k(n)} \\
&= \bigoplus_{\lambda \text{ of } S_n,\, \kappa \text{ of } S_k} V_\lambda^{S_n} \otimes V_\kappa^{S_k} \otimes V_{\lambda,\kappa}^{C(S_n,S_k)}
\end{aligned} \tag{2}
$$

From another point of view we can think of $\vec{x}$ as the fundamental representation $\mathbf{n}$ of $GL(n)$ or $U(n)$ (from physics habit we will refer to the x_i as 'fields'). Vanilla Schur-Weyl duality tells us that if we take k-fold tensor products of the fundamental of $U(n)$ this decomposes exactly into reps κ of both $U(n)$ and S_k, where κ is a partition of k into at most n parts. Since S_n is a subgroup of $U(n)$ we can further decompose each representation $V_\kappa^{U(n)}$ into representations $V_\lambda^{S_n}$ of S_n. These come with exactly the same multiplicity space as we found above.

$$
\begin{aligned}
\left(V_{\mathbf{n}}^{U(n)}\right)^{\otimes k} &= \bigoplus_{\kappa \text{ of } U(n), S_k} V_\kappa^{U(n)} \otimes V_\kappa^{S_k} \\
&= \bigoplus_{\lambda \text{ of } S_n,\, \kappa \text{ of } S_k} V_\lambda^{S_n} \otimes V_{\lambda,\kappa}^{C(S_n,S_k)} \otimes V_\kappa^{S_k}
\end{aligned} \tag{3}
$$

We will now seek to understand the multiplicity space $V_{\lambda,\kappa}^{C(S_n,S_k)}$ from this second point of view, i.e. decomposing $V_\kappa^{U(n)}$ into reps of S_n.

We will often write interchangeably

$$
V_\kappa^{U(n)} \leftrightarrow \kappa\left(V_{\mathrm{nat}}^{\otimes k}\right) \tag{4}
$$

where on the RHS we mean $V_{\mathrm{nat}}^{\otimes k}$ symmetrised by the rep κ of S_k.

In Section 5 below we tabulate examples of the decompositions of $\kappa\left(V_{\mathrm{nat}}^{\otimes k}\right)$ into irreps of S_n; these are worth glancing at to start with.

2 Grading $U(n)$ reps into 'semi-standard' reps of S_n

We can grade $V_\kappa^{U(n)}$ according to which fields x_i appear in each $U(n)$ state of κ. Given the Young diagram κ this corresponds to filling the boxes of κ with the x_i so that they form semi-standard tableaux (i.e. weakly increasing along the rows and strongly increasing down the columns).

2.1 Example: $U(2) \to S_2$ for $k = 2$

As a simple example consider the rep $\begin{array}{|c|}\hline \ \\\hline \ \\\hline\end{array}$ of $U(2)$ for $k = 2$. This has dimension 1 and the single state in this rep is given by the semi-standard tableau

$$
\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \quad \leftrightarrow \quad x_1 \otimes x_2 - x_2 \otimes x_1 \tag{5}
$$

As a rep of S_2 it is antisymmetric

$$
\begin{aligned}
(1)(2)\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} &= \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \\
(12)\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} &= \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} = -\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}
\end{aligned} \tag{6}
$$

so we have

$$V^{U(2)}_{\ydiagram{1,1}} = V^{S_2}_{\ydiagram{1,1}} \tag{7}$$

For the 3-dimensional rep $\ydiagram{2}$ of $U(2)$ we have 3 states in total

$$\young(11) \quad \leftrightarrow \quad x_1 \otimes x_1 \tag{8}$$

$$\young(12) \quad \leftrightarrow \quad x_1 \otimes x_2 + x_2 \otimes x_1 \tag{9}$$

$$\young(22) \quad \leftrightarrow \quad x_2 \otimes x_2 \tag{10}$$

Notice that $\young(12)$ is left invariant by S_2, whereas $\young(11)$ and $\young(22)$ are transformed into each other (forming the natural rep of S_2).

They decompose into S_2 reps as

$$V^{S_2}_{\ydiagram{2}} \quad \leftrightarrow \quad x_1 \otimes x_2 + x_2 \otimes x_1 \tag{11}$$

$$V^{S_2}_{\ydiagram{2}} \quad \leftrightarrow \quad x_1 \otimes x_1 + x_2 \otimes x_2 \tag{12}$$

$$V^{S_2}_{\ydiagram{1,1}} \quad \leftrightarrow \quad x_1 \otimes x_1 - x_2 \otimes x_2 \tag{13}$$

so that we get finally

$$V^{U(2)}_{\ydiagram{2}} = V^{S_2}_{\ydiagram{2}} + \left(V^{S_2}_{\ydiagram{2}} + V^{S_2}_{\ydiagram{1,1}} \right) \tag{14}$$

2.2 General story

How do we generalise this story? Suppose we are given a k-box representation κ of $U(n)$ and a 'field content' of μ_1 x_1's, μ_2 x_2's, $\ldots \mu_n$ x_n's. μ is an *ordered* partition of k into at most n parts, which we will write $\mu \in OP(k,n)$. For example in equation (8) we have $\mu = [2,0]$, in (9) we have $\mu = [1,1]$ and in (10) we have $\mu = [0,2]$. Because these are ordered partitions, we count $[2,0]$ and $[0,2]$ separately.

The number of compatible semi-standard tableaux for a diagram of shape κ is the Kostka number $K_{\kappa,\mu}$. This Kostka number can also be defined as the number of times κ appears in the $U(k)$ tensor product of n totally symmetry $U(k)$ representations $[\mu_1] \otimes [\mu_2] \otimes \cdots \otimes [\mu_n]$. Using the letter g for the Littlewood-Richardson coefficient we write this

$$K_{\kappa,\mu} = g([\mu_1], [\mu_2], \ldots [\mu_n]; \kappa) \tag{15}$$

Given an *unordered* partition M of k into n parts, which we write $M \in P(k,n)$, we define a subset of the ordered partitions $OP_M \subset OP(k,n)$ such that the $\mu \in OP_M$ correspond to M when unordered. For example, if $M = [3,1,0]$ then

$$OP_M = \{[3,1,0], [3,0,1], [1,3,0], [0,3,1], [1,0,3], [0,1,3]\} \tag{16}$$

First non-trivial statement: the semi-standard tableaux corresponding to each set OP_M form a (reducible) representation of S_n, which we shall call $R_{\kappa,M}$. Its size is given by

$$|R_{\kappa,M}| = |OP_M| K_{\kappa,M} \tag{17}$$

Why is this a rep? Consider a state in $V_\kappa^{U(n)}$ with field content $\mu \in OP_M$, i.e. μ_i x_i's for $i \in \{1, \ldots n\}$. If we act on the fields with $\sigma \in S_n$ we will get another semi-standard tableaux with μ_i $x_{\sigma(i)}$'s, or in other words $\mu_{\sigma^{-1}(i)}$ x_i's. The field content $\mu_{\sigma^{-1}}$ is also in OP_M. So if we act on the fields with S_n then we are moved to another state in $V_\kappa^{U(n)}$ with field content also in OP_M.

Thus we haved graded $V_\kappa^{U(n)}$ into reducible reps of S_n

$$V_\kappa^{U(n)} = \bigoplus_{M \in P(k,n)} V_{R_{\kappa,M}}^{S_n} \tag{18}$$

For our $U(2)$ example the S_2 representation

$$R_{\kappa=[2],M=[2,0]} \sim \begin{pmatrix} \boxed{1}\boxed{1} \\ \boxed{2}\boxed{2} \end{pmatrix} \sim \begin{pmatrix} x_1 \otimes x_1 \\ x_2 \otimes x_2 \end{pmatrix} \tag{19}$$

is decomposable into two irreps of S_2.

3 Decomposing the graded reps

Now that we have partially decomposed $V_\kappa^{U(n)}$ into a sum of reducible S_n reps $V_{R_{\kappa,M}}^{S_n}$, we want to further decompose $V_{R_{\kappa,M}}^{S_n}$ into irreps of S_n. We want to be able to write

$$V_\kappa^{U(n)} = \kappa(V_{\text{nat}}^{\otimes k}) = \bigoplus_{M \in P(k,n)} V_{R_{\kappa,M}}^{S_n} = \bigoplus_{M \in P(k,n)} \bigoplus_{\lambda \vdash n} c(R_{\kappa,M}, \lambda) \;\; V_\lambda^{S_n} \tag{20}$$

We will show in a few examples below that the integer coefficients $c(R_{\kappa,M}, \lambda)$ can be described in terms of Littlewood-Richardson coefficients.

First we must define some new quantities. $M \in P(k,n)$ can be written

$$M = [k^{m_k}, (k-1)^{m_{k-1}}, \ldots, 1^{m_1}, 0^{m_0}] \tag{21}$$

where

$$\sum_{p=0}^{k} m_p = n \qquad \text{and} \qquad \sum_{p=0}^{k} p m_p = k \tag{22}$$

The m_p define *both* a partition of k and of n.

We can use the m_p to define the size of the set OP_M, which is just the multinomial coefficient

$$|OP_M| = \frac{n!}{m_0! m_1! \cdots m_k!} \tag{23}$$

Second non-trivial statement: Given a field content $\mu \in OP_M$, m_0 fields do not appear in the semi-standard tableaux. These fields are invariant under the action of an S_{m_0} on them. Thus $R_{\kappa,M}$ will always have the form

$$V_{R_{\kappa,M}}^{S_n} = \sum_{\alpha \vdash n - m_0} d(R_{\kappa,M}, \alpha) \;\; \sum_{\lambda \vdash n} g(\alpha, [m_0]; \lambda) \;\; V_\lambda^{S_n} \tag{24}$$

where $d(R_{\kappa,M}, \alpha)$ is an integer coefficient and $g(\alpha, [m_0]; \lambda)$ is the Littlewood-Richardson coefficient of λ in the symmetric group outer product $\alpha \otimes [m_0]$. Thus the expression (24) has the form of a sum of non-trivial reps of $n - m_0$ in an outer product with the trivial rep of m_0. Dropping the clumsy notation and writing $R_{\kappa,M} \equiv V^{S_n}_{R_{\kappa,M}}$ we write this

$$R_{\kappa,M} = \sum_{\alpha \vdash n-m_0} d(R_{\kappa,M}, \alpha) \quad \alpha \otimes [m_0] \tag{25}$$

where $\otimes$ is the symmetric group outer product.

Thus we have reduced the problem to finding the coefficients $d(R_{\kappa,M}, \alpha)$ of the non-trivial reps α of $n - m_0$. This is equivalent to finding the decomposition of $R_{\kappa,M}$ when $m_0 = 0$.

We have no definitive form for the coefficients $d(R_{\kappa,M}, \alpha)$ but we will dicuss a conjecture in the next section. Examples are tabulated up to $k = 3$ in Table 1 and for $k = 4$ in Table 2.

In the remainder of this section we will give examples of how equation (25) works.

For $\kappa = \square = [1]$ we have n possible fields contents μ, which are in the class given by $M = [1, 0^{n-1}]$ and we have

$$\begin{aligned} R_{[1],[1,0^{n-1}]} &= [1] \otimes [n-1] \\ &= [n] + [n-1, 1] \end{aligned} \tag{26}$$

This is just the natural rep itself, compare with the result (53).

For $\kappa = \square\square$ we can have $M = [2, 0^{n-1}]$

$$\begin{aligned} R_{[2],[2,0^{n-1}]} &= [1] \otimes [n-1] \\ &= [n] + [n-1, 1] \end{aligned} \tag{27}$$

and $M = [1, 1, 0^{n-2}]$

$$\begin{aligned} R_{[2],[1,1,0^{n-2}]} &= [2] \otimes [n-2] \\ &= [n] + [n-1, 1] + [n-2, 2] \end{aligned} \tag{28}$$

The sum of these two reps gives us the result for the decomposition of $V^{U(n)}_{[2]}$, cf. (55),

$$V^{U(n)}_{[2]} = 2[n] + 2[n-1, 1] + [n-2, 2] \tag{29}$$

(note that this is the correct generalisation of our example for $n = 2$ in equation (14), because for $n = 2$ we get $[1] \otimes [1] + [2] \otimes [0] = 2[2] + [1, 1]$).

For $\kappa = \begin{smallmatrix}\square\\\square\end{smallmatrix}$ because of the antisymmetry we can only have field contents given by $M = [1, 1, 0^{n-2}]$ for which

$$\begin{aligned} R_{[1,1],[1,1,0^{n-2}]} &= [1, 1] \otimes [n-2] \\ &= [n-1, 1] + [n-2, 1, 1] \end{aligned} \tag{30}$$

This on its own gives $V^{U(n)}_{[2]}$, cf. (56).

A more complicated example is $\kappa = \begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}$.

For the field content $M = [2, 2, 0^{n-2}]$ we read off the irreps α of S_{n-m_0} from Table 2

$$\begin{aligned} R_{\kappa,M} &= [2] \otimes [n-2] \\ &= [n] + [n-1, 1] + [n-2, 2] \end{aligned} \tag{31}$$

κ	M	$\sum_\alpha d(R_{\kappa,M},\alpha)\ \alpha$	β
$\yng(1)$	$[1,0^{n-1}]$	$\yng(1)$	$\yng(1)$
$\yng(2)$	$[2,0^{n-1}]$	$\yng(1)$	$\yng(1)$
$\yng(2)$	$[1,1,0^{n-2}]$	$\yng(2)$	$\yng(2)$
$\yng(1,1)$	$[1,1,0^{n-2}]$	$\yng(1,1)$	$\yng(1,1)$
$\yng(3)$	$[3,0^{n-1}]$	$\yng(1)$	$\yng(1)$
$\yng(3)$	$[2,1,0^{n-2}]$	$\yng(2)+\yng(1,1)$	$\yng(2)$
$\yng(3)$	$[1,1,1,0^{n-3}]$	$\yng(3)$	$\yng(3)$
$\yng(2,1)$	$[2,1,0^{n-2}]$	$\yng(2)+\yng(1,1)$	$\yng(2)$ OR $\yng(1,1)$
$\yng(2,1)$	$[1,1,1,0^{n-3}]$	$\yng(2,1)$	$\yng(2,1)$
$\yng(1,1,1)$	$[1,1,1,0^{n-3}]$	$\yng(1,1,1)$	$\yng(1,1,1)$

Table 1: tables up to $k = 3$

For $M = [2,1,1,0^{n-3}]$ we get

$$\begin{aligned} R_{\kappa,M} &= [3]\otimes[n-3] \quad + \quad [2,1]\otimes[n-3] \\ &= [n]+[n-1,1]+[n-2,2]+[n-3,3] \\ &\quad +[n-1,1]+[n-2,2]+[n-2,1,1]+[n-3,2,1] \end{aligned} \tag{32}$$

For $M = [1,1,1,1,0^{n-4}]$ we get

$$\begin{aligned} R_{\kappa,M} &= [2,2]\otimes[n-4] \\ &= [n-2,2]+[n-3,2,1]+[n-4,2,2] \end{aligned} \tag{33}$$

If we add all these together we get the result

$$V^{U(n)}_{\yng(2,2)} = 2[n]+3[n-1,1]+4[n-2,2]+[n-2,1,1]+[n-3,3]+2[n-3,2,1]+[n-4,2,2] \tag{34}$$

cf. equation (64).

3.1 $d(R_{\kappa,M},\alpha)$

To find the the $d(R_{\kappa,M},\alpha)$ is equivalent to working out the decomposition of $R_{\kappa,M}$ when $m_0 = 0$.

Consider the S_n character of $R_{\kappa,M}$. Working in the basis of semi-standard tableaux (SST) the character is determined by the SST that are preserved by σ (as only these appear on the diagonal of the matrix for σ)

$$\chi_{R_{\kappa,M}}(\sigma) = \sum \pm\{\text{ SST preserved by } \sigma\} \tag{35}$$

For example, consider the SST for $\kappa = [2,2]$, $M = [2,1,1]$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array} \tag{36}$$

Each of these is preserved by the identity, so $\chi_{R_{[2,2],[2,1,1]}}((1)(2)(3)) = 3$, which is just the dimension. Now consider the action of $\sigma = (23)$. Only the field content of the first SST is preserved

$$(23)\ \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array} = \begin{array}{|c|c|}\hline 1&1\\\hline 3&2\\\hline\end{array} = \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array} \tag{37}$$

so that $\chi_{R_{[2,2],[2,1,1]}}((1)(23)) = 1$. 3-cycles do not preserve the field content for any of the SST, so $\chi_{R_{[2,2],[2,1,1]}}((123)) = 0$. The characters thus fix $R_{[2,2],[2,1,1]}$ as the natural representation of S_3.

The character can also be negative, as in this example

$$(23)\ \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1} 2\\\cline{1-1}\end{array} = -\ \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array} \tag{38}$$

which results in $\chi_{R_{[2,1,1],[2,1,1]}}((1)(23)) = -1$.

From these examples we can see that in general $\chi_{R_{\kappa,M}}(\sigma)$ is only non-zero if σ preserves the field content i.e.

$$\chi_{R_{\kappa,M}}(\sigma) \neq 0 \quad \Rightarrow \quad \sigma \in [S_{m_1} \times S_{m_2} \times \cdots S_{m_k}] \tag{39}$$

where $[\cdots]$ means 'in the conjugacy class of'. This condition can be satisfied if and only if

$$R_{\kappa,M} = \lambda_1 \otimes \lambda_2 \otimes \cdots \lambda_k \tag{40}$$

where for each p λ_p is a (possibly reducible) rep of S_{m_p}.

Note that the condition (39) is not $\Leftarrow$, because of the multiplicity of SST with the same field content. Take for example $\kappa = [3,1]$ and $M = [2,1,1]$ which has $K_{[3,1],[2,1,1]} = 2$, i.e. two valid SST for each field content. The permutation $(23) \in S_3$ preserves the field content of the first SST but the result is a different valid SST

$$(23)\ \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 3\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2\\\cline{1-1}\end{array} \tag{41}$$

Thus $\chi_{R_{[3,1],[2,1,1]}}((23)) = 0$.

4 Relation to plethysm

For $k = 2n$ consider, e.g. $n = 3$, $k = 6$ $\kappa \vdash 2n$ a rep of $U(n)$

$$g([2],[2],[2];\kappa) = \sum_{\lambda \vdash n} c_{\kappa,\lambda}\lambda \tag{42}$$

Here $|OP_M| = 1 = n!/n!$. Decomposing $g([2],[2],[2];\kappa)$ into $\lambda \vdash n$ reps is equivalent to solving the plethysm problem

$$\lambda([2]^{\otimes n}) = \sum_{\kappa \vdash 2n} c_{\kappa,\lambda} \kappa \tag{43}$$

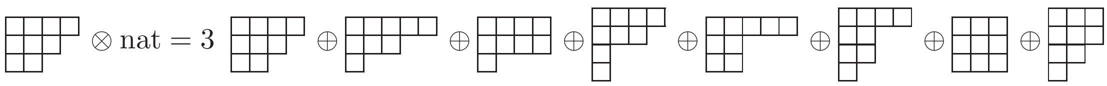
5 Tensor products of the natural

It is very easy to take tensor products of V_{nat} because

$$V_\lambda^{S_n} \otimes V_{\text{nat}}^{S_n} = \bigoplus_{\mu=(\lambda^-)^+} V_\mu^{S_n} \tag{44}$$

Knock a box off λ and then add it back somewhere. $V_\lambda^{S_n}$ itself appears with a multiplicity equal to the number of boxes free to remove, e.g. for $\lambda = [3,2]$ it appears twice, for $\lambda = [2,2,2]$ it appears once.

For example



(45)

This gives us an expansion

$$V_{\text{nat}}^{\otimes k} = \bigoplus_{\lambda \vdash n} V_\lambda^{S_n} \otimes V_\lambda^{P_k(n)} \tag{46}$$

where the dimension of $V_\lambda^{P_k(n)}$, which is the multiplicity of $V_\lambda^{S_n}$, is given by

$$\begin{aligned} \dim V_\lambda^{P_k(n)} &= \operatorname{tr}_{V_{\text{nat}}^{\otimes k}}(P_\lambda) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \left[\chi_{\text{nat}}(\sigma)\right]^k \end{aligned} \tag{47}$$

If we break down $P_k(n) \to \mathbb{C}S_k$ then we can break $V_\lambda^{P_k(n)}$ into reps $V_\kappa^{S_k}$ of S_k so we get

$$\left(V_{\text{nat}}^{S_n}\right)^{\otimes k} = \bigoplus_{\lambda \text{ of } S_n,\, \kappa \text{ of } S_k} V_\lambda^{S_n} \otimes V_\kappa^{S_k} \otimes V_{\lambda,\kappa}^{C(S_n,S_k)} \tag{48}$$

where the dimension of $V_{\lambda,\kappa}^{C(S_n,S_k)}$, which is the multiplicity of $V_\lambda^{S_n} \otimes V_\kappa^{S_k}$, is given by

$$\begin{aligned} \dim V_{\lambda,\kappa}^{C(S_n,S_k)} &= \operatorname{tr}_{V_{\text{nat}}^{\otimes k}}(P_\lambda \otimes P_\kappa) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \chi_\kappa(\tau) \prod_i (\operatorname{tr}_{V_{\text{nat}}}(\sigma^i))^{c_i(\tau)} \end{aligned} \tag{49}$$

where $c_i(\tau)$ is the number of cycles in τ of length i. We have put this formula into a computer to obtain the examples below.

We're often interest in expanding the $V_\lambda^{S_n}$ for a particular κ of S_k, which we'll often write

$$\kappa(V_{\text{nat}}^{\otimes k}) = \sum_{\lambda \vdash n} \dim V_{\lambda,\kappa}^{C(S_n,S_k)} \;\; V_\lambda^{S_n} \tag{50}$$

For example we know that the antisymmetric product of naturals is always

$$[1^k](V_{\text{nat}}^{\otimes k}) \equiv [n-k+1, 1^{k-1}] + [n-k, 1^k] \tag{51}$$

It's also true that for $\kappa(V_{\text{nat}}^{\otimes k})$, where κ is a k-box Young diagram, the only rep of the form $[n-k, *]$ in $\kappa(V_{\text{nat}}^{\otimes k})$ is $[n-k, \kappa]$.

5.1 $k = 1$

$$V_{\mathrm{nat}}^{\otimes 1} = [n] + [n-1,1] \tag{52}$$

$$\ydiagram{1}\,(V_{\mathrm{nat}}^{\otimes 1}) = [n] + [n-1,1] \tag{53}$$

5.2 $k = 2$

$$V_{\mathrm{nat}}^{\otimes 2} = 2[n] + 3[n-1,1] + [n-2,2] + [n-2,1,1] \tag{54}$$

$$\ydiagram{2}\,(V_{\mathrm{nat}}^{\otimes 2}) = 2[n] + 2[n-1,1] + [n-2,2] \tag{55}$$

$$\ydiagram{1,1}\,(V_{\mathrm{nat}}^{\otimes 2}) = [n-1,1] + [n-2,1,1] \tag{56}$$

5.3 $k = 3$

$$V_{\mathrm{nat}}^{\otimes 3} = 5[n] + 10[n-1,1] + 6[n-2,2] + 6[n-2,1,1] + [n-3,3] + 2[n-3,2,1] + [n-3,1,1,1] \tag{57}$$

$$\ydiagram{3}\,(V_{\mathrm{nat}}^{\otimes 3}) = 3[n] + 4[n-1,1] + 2[n-2,2] + [n-2,1,1] + [n-3,3] \tag{58}$$

$$\ydiagram{2,1}\,(V_{\mathrm{nat}}^{\otimes 3}) = [n] + 3[n-1,1] + 2[n-2,2] + 2[n-2,1,1] + [n-3,2,1] \tag{59}$$

$$\ydiagram{1,1,1}\,(V_{\mathrm{nat}}^{\otimes 3}) = [n-2,1,1] + [n-3,1,1,1] \tag{60}$$

5.4 $k = 4$

$$\begin{aligned} V_{\mathrm{nat}}^{\otimes 4} = & 15[n] + 37[n-1,1] + 31[n-2,2] + 31[n-2,1,1] \\ & + 10[n-3,3] + 20[n-3,2,1] + 10[n-3,1,1,1] \\ & + [n-4,4] + 3[n-4,3,1] + 2[n-4,2,2] + 3[n-4,2,1,1] + [n-4,1,1,1,1] \end{aligned} \tag{61}$$

$$\begin{aligned} \ydiagram{4}\,(V_{\mathrm{nat}}^{\otimes 4}) = & 5[n] + 7[n-1,1] + 5[n-2,2] + 2[n-2,1,1] \\ & + 2[n-3,3] + [n-3,2,1] + [n-4,4] \end{aligned} \tag{62}$$

$$\begin{aligned} \ydiagram{3,1}\,(V_{\mathrm{nat}}^{\otimes 4}) = & 2[n] + 7[n-1,1] + 5[n-2,2] + 6[n-2,1,1] \\ & + 2[n-3,3] + 3[n-3,2,1] + [n-3,1,1,1] + [n-4,3,1] \end{aligned} \tag{63}$$

$$\ydiagram{2,2}\,(V_{\mathrm{nat}}^{\otimes 4}) = 2[n] + 3[n-1,1] + 4[n-2,2] + [n-2,1,1] + [n-3,3] + 2[n-3,2,1] + [n-4,2,2] \tag{64}$$

$$\ydiagram{2,1,1}\,(V_{\mathrm{nat}}^{\otimes 4}) = [n-1,1] + [n-2,2] + 3[n-2,1,1] + 2[n-3,2,1] + 2[n-3,1,1,1] + [n-4,2,1,1] \tag{65}$$

$$\ydiagram{1,1,1,1}\,(V_{\mathrm{nat}}^{\otimes 4}) = [n-3,1,1,1] + [n-4,1,1,1,1] \tag{66}$$

κ	M	$\sum_\alpha d(R_{\kappa,M},\alpha)\ \alpha$	β
$\yng(4)$	$[4, 0^{n-1}]$	$\yng(1)$	$\yng(1)$
$\yng(4)$	$[3, 1, 0^{n-2}]$	$\yng(2) + \yng(1,1)$	$\yng(2)$
$\yng(4)$	$[2, 2, 0^{n-2}]$	$\yng(2)$	$\yng(2)$
$\yng(4)$	$[2, 1, 1, 0^{n-3}]$	$\yng(3) + \yng(2,1)$	$\yng(3)$
$\yng(4)$	$[1, 1, 1, 1, 0^{n-4}]$	$\yng(4)$	$\yng(4)$
$\yng(3,1)$	$[3, 1, 0^{n-2}]$	$\yng(2) + \yng(1,1)$	$\yng(2)$ OR $\yng(1,1)$
$\yng(3,1)$	$[2, 2, 0^{n-2}]$	$\yng(1,1)$	$\yng(1,1)$
$\yng(3,1)$	$[2, 1, 1, 0^{n-3}]$	$\yng(3) + 2\yng(2,1) + \yng(1,1,1)$	$\yng(2,1)$
$\yng(3,1)$	$[1, 1, 1, 1, 0^{n-4}]$	$\yng(3,1)$	$\yng(3,1)$
$\yng(2,2)$	$[2, 2, 0^{n-2}]$	$\yng(2)$	$\yng(2)$
$\yng(2,2)$	$[2, 1, 1, 0^{n-3}]$	$\yng(3) + \yng(2,1)$	$\yng(3)$
$\yng(2,2)$	$[1, 1, 1, 1, 0^{n-4}]$	$\yng(2,2)$	$\yng(2,2)$
$\yng(2,1,1)$	$[2, 1, 1, 0^{n-3}]$	$\yng(2,1) + \yng(1,1,1)$	$\yng(1,1,1)$
$\yng(2,1,1)$	$[1, 1, 1, 1, 0^{n-4}]$	$\yng(2,1,1)$	$\yng(2,1,1)$
$\yng(1,1,1,1)$	$[1, 1, 1, 1, 0^{n-4}]$	$\yng(1,1,1,1)$	$\yng(1,1,1,1)$

Table 2: table of $k = 4$

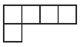
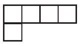
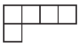
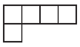
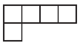
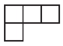
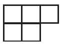
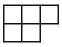
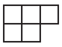
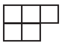
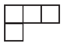
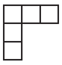
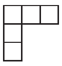
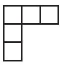
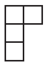
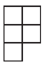
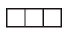
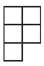
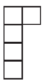
κ	M	β
	$[4,1]$	 OR 
	$[3,2]$	 OR 
	$[3,1,1]$	
	$[2,2,1]$	
	$[2,1,1,1]$	
	$[3,2]$	 OR 
	$[3,1,1]$	
	$[2,2,1]$	
	$[2,1,1,1]$	
	$[3,1,1]$	
	$[2,2,1]$	
	$[2,1,1,1]$	
	$[2,2,1]$	
	$[2,1,1,1]$	
	$[2,1,1,1]$	

Table 3: table of $k = 5$, only non-obvious examples listed