Onshell $SO(2,4)$

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Contents

1 Introduction

Let V_F be the representation of $SO(2,4)$ containing all the fundamental fields $V_F = \{X, \partial_\mu X, \partial_{\mu_1}\partial_{\mu_2} X, \dots\}$. We want to understand how to decompose arbitrary tensor products $V_F^{\otimes n}$ into representations Λ of $SO(2,4)$ and λ of S_n.

$$V_F^{\otimes n} = \sum_{\Lambda}\sum_{\lambda\vdash n} \;\mathrm{mult}(\Lambda,\lambda)\; V_\Lambda^{SO(2,4)} \;\otimes\; V_\lambda^{S_n} \tag{1}$$

The irrep labels of $SO(2,4)$ are $\Lambda = \{\Delta, j_L, j_R\}$ where $\Delta \in \mathbb{N}\cup\{0\}$ and $j_L, j_R \in \frac{1}{2}\mathbb{N}\cup\{0\}$.[1]

We use an oscillator construction to build representations of $SO(2,4)$. The vacuum $|0\rangle$ corresponds to $X^{\otimes n}$ and the oscillator $a^\dagger_{i\mu}$ acting on the vacuum $a^\dagger_{i\mu}|0\rangle$ corresponds to the derivative ∂_μ acting on the ith site.

To get highest weight states (HWSs) we take linear combinations $A^\dagger_{h\mu} = J^i_h a^\dagger_{i\mu}$ corresponding to the hook representation $H = [n-1, 1]$ of S_n. h transforms in V_H.

The HWSs are given with $SO(4)$ indices by

$$A^\dagger_{h_1\mu_1}\cdots A^\dagger_{h_k\mu_k}|0\rangle \tag{2}$$

or alternatively with $SU(2)_L \times SU(2)_R$ indices

$$A^\dagger_{h_1\alpha_1\dot\alpha_1}\cdots A^\dagger_{h_k\alpha_k\dot\alpha_k}|0\rangle \tag{3}$$

2 The offshell case

2.1 $GL(4)$ offshell operator

We want to organise the operators

$$A^\dagger_{h_1\mu_1}\cdots A^\dagger_{h_k\mu_k}|0\rangle \tag{4}$$

into irreps of $SO(4)$ and S_n. A first step is to organise them into irreps of $GL(4)$ and $GL(d_H)$.

We can organise the $SO(4)$ indices μ_i in terms of $GL(4)$ reps K with k boxes and ≤ 4 rows. These reduce to $SO(4)$ reps in a procedure we will describe later. If $V_{\mathbf{4}}^{GL(4)}$ is the fundamental of $GL(4)$ then Schur-Weyl duality tells us that

$$\left(V_{\mathbf{4}}^{GL(4)}\right)^{\otimes k} = \bigoplus_{K\in P(k,4)} V_K^{GL(4)} \otimes V_K^{S_k} \tag{5}$$

We have summed over partitions K in $P(k,4)$ with k boxes and ≤ 4 rows, which correspond both to representations of $GL(4)$ and S_k. The corresponding Clebsch-Gordan coefficient is

$$C^{\mu_1\cdots\mu_k}_{K,M_K,a_K} \tag{6}$$

[1] Note that for derivatives of scalars, if j_L is integer j_R must be too, and similarly if j_L is half-integer. We also have $\Delta - n = k \geq 2j_L$ and $\Delta - n = k \geq 2j_R$.

M_K labels the $GL(4)$ state in $V_K^{GL(4)}$ and a_K the S_k state in $V_K^{S_k}$.

Similarly we can organise the V_H indices h_i in terms of $GL(d_H)$ reps K' with k boxes and $\leq d_H$ rows. These reduce to S_n reps in a procedure we will describe later. By Schur-Weyl duality

$$(V_H)^{\otimes k} = \bigoplus_{K' \in P(k, d_H)} V_{K'}^{GL(d_H)} \otimes V_{K'}^{S_k} \tag{7}$$

which Clebsch-Gordan coefficient

$$C^{h_1 \cdots h_k}_{K', M'_{K'}, a'_{K'}} \tag{8}$$

Because the $A^\dagger_{h_i \mu_i}$ commute, the overall operator transforms in $\mathrm{Sym}((V_{\mathbf{4}} \otimes V_H)^{\otimes k})$. As discussed in Appendix Section C.2, this triviality under S_k forces $K = K'$ and we must sum over the S_k states to get

$$|K, M_K, M'_K\rangle = \sum_{a_K} C^{\mu_1 \cdots \mu_k}_{K, M_K, a_K} \; C^{h_1 \cdots h_k}_{K, M'_K a_K} \; A^\dagger_{h_1 \mu_1} \cdots A^\dagger_{h_k \mu_k} |0\rangle \tag{9}$$

Since $K \in P(k, 4)$ it is clear that the rep K organising the hook indices h_i can't have more than 4 rows.

This can further be transformed into a state of the symmetric rep $[k]$ of $GL(4d_H)$ with the Clebsch-Gordan coefficient

$$\big|[k], M_{[k]}\big\rangle = \sum_{K \in P(k,4)} \sum_{M_K} \sum_{M'_K} C^{K, M_K, M'_K}_{[k], M_{[k]}} \, |K, M_K, M'_K\rangle \tag{10}$$

2.2 Decomposing $GL(d_H)$ reps K into S_n reps

We can further decompose an irrep K of $GL(d_H)$ into irreps λ of S_n

$$V_K^{GL(d_H)} = \bigoplus_{\lambda \in P(n)} V_\lambda^{S_n} \otimes V_{\lambda, K} \tag{11}$$

This gives an overall decomposition

$$\left(V_H^{S_n}\right)^{\otimes k} = \bigoplus_{\lambda \in P(n), K \in P(k)} V_\lambda^{S_n} \otimes V_K^{S_k} \otimes V_{\lambda, K} \tag{12}$$

These reps appear with a multiplicity space $V_{\lambda, K}$ which we label with τ in the Clebsch-Gordan

$$C^{h_1 \cdots h_k}_{\lambda, a_\lambda, K, a_K, \tau} \tag{13}$$

For example for $K = \square\!\square$ of $GL(d_H)$ we have

$$V_{\square\!\square}^{GL(d_H)} = \square\!\square\left(V_H^{\circ 2}\right) = [n] \oplus [n-1, 1] \oplus [n-2, 2] \tag{14}$$

2.3 Decomposing $GL(4)$ reps K into $SO(4)$ reps

$SO(4)$ is a subgroup of $GL(4)$, so representations K of $GL(4)$ are also representations of $SO(4)$. However under $SO(4)$ reps K of $GL(4)$ may be reducible. In general we will have a decomposition

$$V_K^{GL(4)} = \bigoplus_{\Lambda} V_\Lambda^{SO(4)} \otimes V_{K, \Lambda} \tag{15}$$

We have summed over reps Λ of $SO(4)$ contained inside K, which occur with a multiplicity space $V_{K,\Lambda}$ whose dimension is in $\{0, 1\}$. An $SO(4)$ rep is a 2-row Young diagram with row-lengths given by the $SU(2)_L \times SU(2)_R$ spins

$$\Lambda = [j_L + j_R, |j_L - j_R|] \tag{16}$$

$SO(4)$ has the two invariant tensors

$$\eta^{\mu_1 \mu_2} \quad \text{and} \quad \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \tag{17}$$

In $GL(4)$ language these appear in

$$\square\!\square \quad \text{and} \quad \begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\\[-4pt]\square\end{array} \tag{18}$$

Reducing the k-boxed 4-row $GL(4)$ Young diagram K to Λ is a matter of taking account of the two invariant tensors η and ϵ. First we remove all possible even partitions $2T$ from K, corresponding to η contractions (products of $\yng(2)$ give even partitions). Then project the remaining 4-row Young diagram Λ' with π to an $SO(4)$ 2-row Young diagram Λ; this removes the ϵ tensor. Thus

$$K = \bigoplus_{\Lambda} \dim V_{K,\Lambda} \;\; \Lambda = \bigoplus_{2T,\Lambda'} g(2T,\Lambda';K) \;\; \pi(\Lambda') \tag{19}$$

We have summed over even partitions $2T$ which correspond to contractions η.[2] The Λ' are then projected to $SO(4)$ reps Λ. A complete list of these projections is given in Appendix Section A.

For example

$$\begin{aligned}
\yng(2,2) &= g\left(\mathbf{1},\yng(2,2);\yng(2,2)\right)\pi\left(\yng(2,2)\right) \oplus g\left(\yng(2),\yng(2);\yng(2,2)\right)\pi\left(\yng(2)\right) \oplus g\left(\yng(2,2),\mathbf{1};\yng(2,2)\right)\pi\left(\mathbf{1}\right) \\
&= \pi\left(\yng(2,2)\right) \oplus \pi\left(\yng(2)\right) \oplus \pi\left(\mathbf{1}\right) \\
&= \yng(2,2) \oplus \yng(2) \oplus \mathbf{1}
\end{aligned} \tag{20}$$

which works dimensionally as $20 = 10 + 9 + 1$.

There is however a complication: sometimes $\pi(\Lambda')$ projects to a representation Λ of $SO(4)$ that appears with a sign that cancels another $SO(4)$ rep, for example

$$\begin{aligned}
\yng(2,2,1,1) &= g\left(\mathbf{1},\yng(2,2,1,1);\yng(2,2,1,1)\right)\pi\left(\yng(2,2,1,1)\right) \oplus g\left(\yng(2),\yng(2,1,1);\yng(2,2,1,1)\right)\pi\left(\yng(2,1,1)\right) \oplus g\left(\yng(2,2),\yng(1,1);\yng(2,2)\right)\pi\left(\yng(1,1)\right) \\
&= \pi\left(\yng(2,2,1,1)\right) \oplus \pi\left(\yng(2,1,1)\right) \oplus \pi\left(\yng(1,1)\right) \\
&= -\yng(2) \oplus \yng(2) \oplus \yng(1,1) \\
&= \yng(1,1)
\end{aligned} \tag{21}$$

We don't want some operators to appear with a negative sign and cancel other operators. Thus we redefine 'effective' coefficients

- $\tilde{\pi}(\Lambda')$ such that $\tilde{\pi}(\Lambda') = \pi(\Lambda')$ when the sign "makes sense". Otherwise $\tilde{\pi}(\Lambda') = 0$. Note that $\tilde{\pi}(\Lambda')$ is always either 0 or 1. See Appendix Section B for a full description of $\tilde{\pi}$.
- $\tilde{g}(2T,\Lambda';K)$ is zero for the reps that get cancelled by the $\pi(\Lambda')$ which don't make sense. Note that $\tilde{g}(2T,\Lambda';K) \leq g(2T,\Lambda';K)$.

Thus we get

$$K = \bigoplus_{\Lambda} \dim V_{K,\Lambda} \;\; \Lambda = \bigoplus_{2T,\Lambda'} \tilde{g}(2T,\Lambda';K) \;\; \tilde{\pi}(\Lambda') \tag{22}$$

where everything appears with a positive sign.

So for example

$$\tilde{\pi}\left(\yng(2,2,1,1)\right) = 0 \tag{23}$$

and

$$\tilde{g}\left(\yng(2),\yng(2,1,1);\yng(2,2,1,1)\right) = 0 \tag{24}$$

which gives

$$\begin{aligned}
\yng(2,2,1,1) &= \tilde{g}\left(\mathbf{1},\yng(2,2,1,1);\yng(2,2,1,1)\right)\tilde{\pi}\left(\yng(2,2,1,1)\right) \oplus \tilde{g}\left(\yng(2),\yng(2,1,1);\yng(2,2,1,1)\right)\tilde{\pi}\left(\yng(2,1,1)\right) \oplus \tilde{g}\left(\yng(2,2),\yng(1,1);\yng(2,2)\right)\tilde{\pi}\left(\yng(1,1)\right) \\
&= \tilde{\pi}\left(\yng(2,2,1,1)\right) \oplus \tilde{\pi}\left(\yng(1,1)\right) \\
&= \yng(1,1)
\end{aligned} \tag{25}$$

[2]Since K has ≤ 4 rows, so must anything that is used to build it using the LR rule, e.g. both $2T$ and Λ'.

Another example with a much more complicated cancellation

$$
\begin{aligned}
\ydiagram{3,3,3,1} &= \pi\left(\ydiagram{3,3,3,1}\right) \oplus \pi\left(\ydiagram{3,3,2}\right) \oplus \pi\left(\ydiagram{3,2,1,1}\right) \oplus \pi\left(\ydiagram{3,2,1}\right) \oplus \pi\left(\ydiagram{3,1,1,1}\right) \oplus \pi\left(\ydiagram{2,1,1}\right) \oplus \pi\left(\ydiagram{1,1,1,1}\right) \\
&= \ydiagram{3,3} \oplus -\ydiagram{3,3} \oplus 0 \oplus 0 \oplus 0 \oplus \ydiagram{2} \oplus \mathbf{1} \\
&= \ydiagram{2} \oplus \mathbf{1}
\end{aligned}
\tag{26}
$$

Dimensionally this works $10 = 9 + 1$. In the effective description we would have

$$
\tilde{\pi}\left(\ydiagram{3,3,3,1}\right) = 0 \quad \text{and} \quad \tilde{\pi}\left(\ydiagram{3,3,2}\right) = 0 \tag{27}
$$

2.4 Going backwards

Suppose on the other hand we are given k and Λ and we want to work out not only the $GL(4)$ rep K but also the structure of the tensor K and how it contains the two invariant $SO(4)$ tensors

$$
\eta_{\mu_1\mu_2} \quad \text{and} \quad \epsilon_{\mu_1\mu_2\mu_3\mu_4} \tag{28}
$$

This is extremely important because in the onshell case we want to apply the equations of motion whenever η appears.

The procedure is as follows:

- Take Λ and look up its inverses under the $\tilde{\pi}$ projection $\tilde{\pi}^{-1}(\Lambda) = \{\Lambda'\}$. These inverses are listed in Appendix Section B.1.
- Given a Λ' with k' boxes this defines a $GL(4)$ tensor with no contractions η. Λ' may now contain ϵ's.
- Next we need to add t contractions η to Λ' to make it up to a $GL(4)$ Young diagram K with $k = k' + 2t$ boxes. We do this by $GL(4)$-tensoring all even partitions $2T$ with $2t$ boxes and at most 4 rows with Λ' to get K, as long as the effective coupling is non-zero $\tilde{g}(2T, \Lambda'; K) \geq 1$.

For a given k and Λ this will give a list of $GL(4)$ tensors K

$$
\{K\} = \sum_{k'} \bigoplus_{\Lambda' \vdash k', 2T \vdash k-k'} \tilde{g}(2T, \Lambda'; K)\, \delta(\Lambda = \tilde{\pi}(\Lambda')) \tag{29}
$$

This list is entirely positive and contains no cancellations. Looking forward to the $SU(2)_L \times SU(2)_R$ section, $\Lambda_L \otimes \Lambda_R = \{K\}$ as defined here.

2.5 The explicit decomposed operator

We will now explicitly decompose the $GL(4)$ tensor, separating the t η contractions from the Λ' tensor that projects to Λ with $\tilde{\pi}$.

To do this we first want to effect for $W = V_{\mathbf{4}} \otimes V_H$

$$
\mathrm{Sym}(W^{\otimes k}) \to V_{\mathbf{4}}^{\otimes 2t} \otimes V_{\mathbf{4}}^{\otimes k'} \otimes V_H^{\otimes 2t} \otimes V_H^{\otimes k'} \tag{30}
$$

See Appendix Section C.3.

We get

$$
\begin{aligned}
&|K, M_K, M'_K, H, \Lambda', \tau\rangle \\
&= \sum_{M_H, M'_H} \sum_{M_{\Lambda'}, M'_{\Lambda'}} \sum_{a_H, a_{\Lambda'}} C^{\tau, M_K}_{M_H, M_{\Lambda'}} C^{\tau, M'_K}_{M'_H, M'_{\Lambda'}} C^{\mu_1 \cdots \mu_{2t}}_{H, M_H, a_H} C^{\mu_{2t+1} \cdots \mu_k}_{\Lambda', M_{\Lambda'}, a_{\Lambda'}} C^{h_1 \cdots h_{2t}}_{H, M'_H, a_H} C^{h_{2t+1} \cdots h_k}_{\Lambda', M'_{\Lambda'}, a_{\Lambda'}} \ A^\dagger_{h_1\mu_1} \cdots A^\dagger_{h_k\mu_k} |0\rangle
\end{aligned}
\tag{31}
$$

K is a k-box, 4-row rep with $GL(4)$ state M_K and $GL(d_H)$ state M'_K. $H \in P(2t, 4)$ and $\Lambda' \in P(k', 4)$. τ runs over $g(H, \Lambda'; K)$ for the $GL(4)$ tensor product $H \circ \Lambda' = \bigoplus_K g(H, \Lambda'; K)\, K$.

Now we will butcher this operator for the $GL(4)$ to $SO(4)$ decomposition, paying attention to the interplay between $V_{\mathbf{4}}$ and V_H. We will

- Replace the $V_{\mathbf{4}}^{\otimes 2t}$ tensor $C^{\mu_1\cdots\mu_{2t}}_{H,M_H,a_H}$ by $\eta^{\mu_1\mu_2}\cdots\eta^{\mu_{2t-1}\mu_{2t}}$. This removes the M_H multiplicity.

- This forces an $S_2^t \ltimes S_t$ symmetry on the corresponding $V_H^{\otimes 2t}$ tensor $C^{h_1\cdots h_{2t}}_{H,M'_H,a_H}$. This can be seen most simply if we define $S^\dagger_{h_1h_2} \equiv \eta^{\mu_1\mu_2}A^\dagger_{h_1\mu_1}A^\dagger_{h_1\mu_1}$ and we see that $C^{h_1\cdots h_{2t}}_{H,M'_H,a_H}$ is contracted with $S^\dagger_{h_1h_2}\cdots S^\dagger_{h_{2t-1}h_{2t}}$. As discussed below in Section 2.5.1 this symmetry forces H to have only even rows $H=2T$. It also kills the a_{2T} multiplicity to leave just the M'_{2T} multiplicity.

- $\tilde{\tau}$ now runs over the effective multiplicity $\tilde{g}(2T,\Lambda';K)$ instead of $g(2T,\Lambda';K)$. Because K is a 4-row tensor, $2T$ and Λ' and their product can only have 4 rows.

- Λ' and its $GL(4)$ state $M_{\Lambda'}$ project down to the $SO(4)$ rep Λ and $SO(4)$ state M_Λ with the projection $\tilde{\pi}$. We will write this $\tilde{\Pi}^{\Lambda',M_{\Lambda'}}_{\Lambda,M_\Lambda}$. There is no multiplicity here.

This results in an operator

$$
\begin{aligned}
&|K, M'_K, 2T, \Lambda', \Lambda, M_\Lambda, \tilde{\tau}\rangle \\
&= \sum_{a_{\Lambda'}} C^{\tilde{\tau},M'_K}_{M'_{2T},M'_{\Lambda'}}\ \tilde{\Pi}^{\Lambda',M_{\Lambda'}}_{\Lambda,M_\Lambda}\ C^{\mu_{2t+1}\cdots\mu_k}_{\Lambda',M_{\Lambda'},a_{\Lambda'}} C^{h_1\cdots h_{2t}}_{2T,M'_{2T}}\ C^{h_{2t+1}\cdots h_k}_{\Lambda',M'_{\Lambda'},a_{\Lambda'}}\ \ S^\dagger_{h_1h_2}\cdots S^\dagger_{h_{2t-1}h_{2t}} A^\dagger_{h_{2t+1}\mu_{2t+1}}\cdots A^\dagger_{h_k\mu_k}|0\rangle
\end{aligned} \tag{32}
$$

To get the S_n rep λ we must further decompose the $GL(d_H)$ state M'_K of K along the lines of Section 2.2.

2.5.1 The $S_2^t \ltimes S_t$ reduction

A note on the coefficients $C^{h_1h_2\cdots h_{2t}}_{2T,M'_{2T}}$. We can first decompose the tensor product $V_H^{\otimes 2t}$ in the obvious way into irreps of $GL(V_H)\otimes S_{2t}$. This is done with coefficients:

$$
C^{h_1\cdots h_{2t}}_{H,M'_H,a_H} \tag{33}
$$

Now the symmetry conditions on the indices are invariance under $S_2^t \ltimes S_t$, i.e picking the trivial rep of this group which comes from $(\mathbf{1},\mathbf{1})$ of S_2^t and S_t. The semi-direct product is a subgroup of S_{2t}. We can decompose the states (H,a_H) of S_{2t} into irreps of the semidirect product subgroup. We need to pick the trivial irrep. of this semi-direct product. So we have a branching coefficient

$$
C^{(\mathbf{1},\mathbf{1})_{SD}}_{H,a_H} = \delta(H,2T)C^{(\mathbf{1},\mathbf{1})_{SD}}_{2T,a_{2T}} \tag{34}
$$

In other words the branching coefficient is zero unless $H=2T$. So we have a decomposition

$$
C^{h_1h_2\cdots h_{2t}}_{2T,M'_{2T}} = C^{h_1\cdots h_{2t}}_{2T,M'_{2T},a_{2T}} C^{(\mathbf{1},\mathbf{1})_{SD}}_{2T,a_{2T}} \tag{35}
$$

There is a counting check on the statement that the rep. of S_{2t} induced from the trivial of $S_2^t \ltimes S_t$ is the direct sum of even YD. The order of the semi-direct product group is $2^t t!$. The rep. induced from the trivial has dimension $\frac{(2t)!}{t!2^t}$. We have checked, in examples (as reported in the Appendix of note-EOM7.tex) that

$$
\frac{(2t)!}{t!2^t} = \sum_{2T} d_{2T} \tag{36}
$$

Note that the multiplicity of the rep. H of S_{2t} in the induction of the trivial of the subgroup $S_2^t \ltimes S_t$ is the same as the multiplicity of the trivial of the subgroup in the restriction of H to the subgroup. This induction-restriction duality is Frobenius duality.

The flip Schur-Weyl of this dimension formula is

$$
\mathrm{Dim}_{\frac{d_H(d_H+1)}{2}}[t] = \sum_{2T\in P(2t)} \mathrm{Dim}_{d_H} 2T \tag{37}
$$

2.6 $SO(4)$ counting

For a given $SO(4)$ rep Λ and dimension $\Delta = n + k$ we have using the final operator (32) and (29)

$$\mathrm{mult}(\Lambda,\Delta) = \sum_{K\in P(k,4)} \sum_{k'} \bigoplus_{\Lambda'\vdash k', 2T\vdash k-k'} \tilde{g}(2T,\Lambda';K)\, \delta(\Lambda = \tilde{\pi}(\Lambda'))\;\; \mathrm{Dim}_{d_H} K \tag{38}$$

Refining to a specific S_n rep λ using Section 2.2 we get

$$\mathrm{mult}(\Lambda,\Delta,\lambda) = \sum_{K\in P(k,4)} \sum_{k'} \sum_{\Lambda'\vdash k', 2T\vdash k-k'} \tilde{g}(2T,\Lambda';K)\, \delta(\Lambda = \tilde{\pi}(\Lambda'))\;\; \mathrm{mult}(V_H^{\otimes k}, \lambda\otimes K) \tag{39}$$

We prove these formulae below using $SU(2)_L \times SU(2)_R$ language.

2.7 From $SO(4)$ to $SU(2)_L \times SU(2)_R$

An alternative way of getting the list of $GL(4)$ reps K from k and the $SO(4)$ rep Λ is to take the inner product of the two corresponding $GL(2)_L \times GL(2)_R$ reps. This is more straightforward, but we lose the explicit decomposition of K into η's and ϵ's. This is because the two invariant tensors of $SU(2)_L \times SU(2)_R$ $\epsilon^{\alpha_1\alpha_2}$ and $\epsilon^{\dot\alpha_1\dot\alpha_2}$ don't distinguish η from ϵ. The $SO(4)$ tensors are expressed as

$$\begin{aligned} \eta^{\mu_1\mu_2} a_{i_1\mu_1} a_{i_2\mu_2} &= \epsilon^{\alpha_1\alpha_2}\epsilon^{\dot\alpha_1\dot\alpha_2} a_{i_1\alpha_1\dot\alpha_1} a_{i_2\alpha_2\dot\alpha_2} \\ \epsilon^{\mu_1\mu_2\mu_3\mu_4} a_{i_1\mu_1} a_{i_2\mu_2} a_{i_3\mu_3} a_{i_4\mu_4} &= \epsilon^{\alpha_1\alpha_2}\epsilon^{\dot\alpha_1\dot\alpha_3}\epsilon^{\alpha_3\alpha_4}\epsilon^{\dot\alpha_2\dot\alpha_4} a_{i_1\alpha_1\dot\alpha_1} a_{i_2\alpha_2\dot\alpha_2} a_{i_3\alpha_3\dot\alpha_3} a_{i_4\alpha_4\dot\alpha_4} \end{aligned} \tag{40}$$

It is however much easier to understand the counting from a $SU(2)_L \times SU(2)_R$ perspective.

2.8 $SU(2)_L \times SU(2)_R$ offshell operator

For $SU(2)_L \times SU(2)_R$ we are organising

$$A^\dagger_{h_1\alpha_1\dot\alpha_1} \cdots A^\dagger_{h_k\alpha_k\dot\alpha_k}|0\rangle \tag{41}$$

We can organise the $SU(2)_L$ indices α_i with a $GL(2)_L$ rep $\Lambda_L = [t_L + 2j_L, t_L]$ and the $SU(2)_R$ indices $\dot\alpha_i$ with a $GL(2)_R$ rep $\Lambda_R = [t_R + 2j_R, t_R]$. These numbers satisfy $2t_L + 2j_L = 2t_L + 2j_L = k$ so that Λ_L and Λ_R both contain k boxes.

We proceed for the $GL(2)_L \times GL(2)_R$ tensors as for $GL(4)$

$$C^{\alpha_1\cdots\alpha_k}_{\Lambda_L,M_L,a_L}\;\; C^{\dot\alpha_1\cdots\dot\alpha_k}_{\Lambda_R,M_R,a_R}\;\; C^{h_1\cdots h_k}_{\lambda,a_\lambda,\kappa,a_\kappa,\tau}\;\; A^\dagger_{h_1\alpha_1\dot\alpha_1} \cdots A^\dagger_{h_k\alpha_k\dot\alpha_k}|0\rangle \tag{42}$$

The $A^\dagger_{h_i\alpha_i\dot\alpha_i}$ all commute, so the overall operator transforms in the trivial $[k]$ of S_k. Thus we combine the free S_k indices of this operator with an S_k Clebsch-Gordan

$$\begin{aligned} &\hat{\mathcal{O}}[\Lambda_L, M_L, \Lambda_R, M_R, \lambda, a_\lambda, \{\tau,\kappa,\hat\tau\}] \\ &= C^{\Lambda_L,a_L;\Lambda_R,a_R;\kappa,a_\kappa}_{[k],\hat\tau} C^{\alpha_1\cdots\alpha_k}_{\Lambda_L,M_L,a_L}\;\; C^{\dot\alpha_1\cdots\dot\alpha_k}_{\Lambda_R,M_R,a_R}\;\; C^{h_1\cdots h_k}_{\lambda,a_\lambda,\kappa,a_\kappa,\tau}\;\; A^\dagger_{h_1\alpha_1\dot\alpha_1} \cdots A^\dagger_{h_k\alpha_k\dot\alpha_k}|0\rangle \end{aligned} \tag{43}$$

$\hat\tau$ labels the multiplicity of $[k]$ in the S_k tensor product $\Lambda_L \otimes \Lambda_R \otimes \kappa$, or alternatively the number of times κ appears in the S_k tensor product

$$\Lambda_L \otimes \Lambda_R = \sum_\kappa C(\Lambda_L,\Lambda_R,\kappa)\;\; \kappa \tag{44}$$

It is a rule from [1] that the inner product of two two-row reps gives reps with at most four rows. Thus κ has at most 4 rows.

2.8.1 $GL(4)$ as a $GL(2)_L \times GL(2)_R$ product

We can of course convert between $GL(2)_L \times GL(2)_R$ and $GL(4)$, noticing that $2 \times 2 = 4$.

Applying this to the tensor products we see that the 4-row κ in equation (44) is identified with the $GL(4)$ rep K.

Thus to get K from k and Λ, we find the corresponding $GL(2)_L \times GL(2)_R$ reps Λ_L and Λ_R and take their inner product.

2.9 Offshell counting

We focus on the question:

- Given an $SO(2,4)$ rep $(\Delta = n + k, j_L, j_R)$ and an S_n rep λ, how many HWSs are there?

This is most easily answered from the $SU(2)_L \times SU(2)_R$ point of view. Considering the operator (43), we just sum over the $\{\tau, K, \hat{\tau}\}$ multiplicity labels

$$\mathrm{mult}(\Delta, j_L, j_R, \lambda) = \sum_{K \vdash k} C(\Lambda_L, \Lambda_R, K)\ \mathrm{mult}(V_H^{\otimes k}, \lambda \otimes K) \tag{45}$$

where $\hat{\tau}$ runs over the $C(\Lambda_L, \Lambda_R, K)$ times K appears in $\Lambda_L \otimes \Lambda_R$ and τ runs over the $\mathrm{mult}(V_H^{\otimes k}, \lambda \otimes K)$ times $\lambda \otimes K$ appears in $V_H^{\otimes k}$.

More readably we could write this

$$\mathrm{mult}(\Delta, j_L, j_R, \lambda) = \text{ number of times } \lambda \text{ appears in } [\Lambda_L \otimes \Lambda_R]\,(V_H^{\otimes k}) \tag{46}$$

Given the relation between the inner product and $SO(4)$ tensors we can also write this in $SO(4)$ language

$$\mathrm{mult}(\Delta, \Lambda, \lambda) = \sum_{k'} \sum_{\Lambda' \vdash k'} \delta(\Lambda = \tilde{\pi}(\Lambda')) \text{ number of times } \lambda \text{ appears in } \left[\left[\tfrac{k-k'}{2} \right] (\yng(2)^{\circ \frac{k-k'}{2}}) \circ_4 \Lambda' \right] (V_H^{\otimes k}) \tag{47}$$

$$= \sum_{k'} \sum_{\Lambda' \vdash k'} \sum_{2T \vdash k-k'} \sum_{K \vdash k} \delta(\Lambda = \tilde{\pi}(\Lambda'))\, \tilde{g}(2T, \Lambda'; K)\, \mathrm{mult}(V_H^{\otimes k}, \lambda \otimes K) \tag{48}$$

where $2T$ are even partitions; we remember that the tensor products $\circ_4$ and $\tilde{g}$ only allow K with 4 rows; it is also only an effective tensor product.

2.10 Offshell character expansion and proof of counting

Below we will focus on doing the decomposition (1) in terms of $SO(2,4)$ characters. If $\chi_F(s,x,y)$ is the character of V_F then

$$[\chi_F(s,x,y)]^n = \sum_{\Delta, j_L, j_R} \sum_{\lambda \vdash n} \mathrm{mult}(\Delta, j_L, j_R, \lambda)\ \ d_\lambda\, \chi_{\Delta, j_L, j_R}(s,x,y) \tag{49}$$

The offshell character of V_F, all the descendants of X, is

$$\chi_F(s,x,y) = \chi_{1,0,0} = Ps \tag{50}$$

P accounts for all the descendants with derivatives

$$P = \frac{1}{(1-sxy)(1-sx^{-1}y)(1-sxy^{-1})(1-sx^{-1}y^{-1})} \tag{51}$$

For a general $SO(2,4)$ irrep

$$\chi_{\Delta, j_L, j_R}(s,x,y) = Ps^{\Delta} \chi_{j_L}(X) \chi_{j_R}(Y) \tag{52}$$

where $\Delta = n + k$, where k is the number of derivatives for the highest weight, and $X = \mathrm{diag}(x, x^{-1}) \in SU(2)$. Since X is in $SU(2)$ we can remove columns of length two when we work out the character, e.g.

$$\chi_{\yng(5,3)}(X) = \chi_{\yng(2)}(X) \tag{53}$$

As we worked out previously in Section 7 of `sl2diag.dvi` and Section 2 of `note-EOM.dvi` by expanding P^{n-1} in terms of V_H

$$\begin{aligned}
\chi_F^n &= [Ps]^n \\
&= Ps^n \sum_{k=0}^{\infty} s^k \sum_{\Lambda_L, \Lambda_R, \Lambda_2 \vdash k} \sum_{\lambda \vdash n} d_\lambda\, \mathrm{mult}(V_H^{\otimes k}, \lambda \otimes \Lambda_2) C(\Lambda_L, \Lambda_R, \Lambda_2) \chi_{\Lambda_L}(X) \chi_{\Lambda_R}(Y) \\
&= Ps^n \sum_{k, j_L, j_R = 0}^{\infty} s^k \chi_{j_L}(X) \chi_{j_R}(Y) \sum_{\lambda \vdash n} d_\lambda \\
&\quad \sum_{\Lambda_2 \vdash k} \mathrm{mult}(V_H^{\otimes k}, \lambda \otimes \Lambda_2)\ \ C\left(\Lambda_L = \left[\tfrac{k}{2} + j_L, \tfrac{k}{2} - j_L\right], \Lambda_R = \left[\tfrac{k}{2} + j_R, \tfrac{k}{2} - j_R\right], \Lambda_2\right)
\end{aligned} \tag{54}$$

To make life simpler write $\Lambda_L = \{k, j_L\}$ for the $SU(2)$ 2-row Young diagram with k boxes corresponding to the spin j_L rep.

$$\Lambda_L = \left[\tfrac{k}{2} + j_L, \tfrac{k}{2} - j_L\right] \equiv \{k, j_L\} \sim [2j_L] \tag{55}$$

where $[2j_L]$ is the single-row Young diagram with $2j_L$ boxes, corresponding to the spin j_L rep.

This result matches with our goal (49)

$$\mathrm{mult}(\Delta = n+k, j_L, j_R, \lambda) = \sum_{\Lambda_2 \vdash k} \mathrm{mult}(V_H^{\otimes k}, \lambda \otimes \Lambda_2)\;\; C\left(\Lambda_L = \{k, j_L\}, \Lambda_R = \{k, j_R\}, \Lambda_2\right) \tag{56}$$

To get the overall multiplicity of the $SO(2,4)$, ignoring the S_n rep, we sum over the $\lambda \vdash n$,

$$\begin{aligned}\mathrm{mult}(\Delta = n+k, j_L, j_R) &= \sum_{\lambda \vdash n} d_\lambda\, \mathrm{mult}(\Delta = n+k, j_L, j_R, \lambda) \\ &= \sum_{\Lambda_2 \vdash k} \dim_{n-1} \Lambda_2 \;\; C\left(\Lambda_L = \{k, j_L\}, \Lambda_R = \{k, j_R\}, \Lambda_2\right)\end{aligned} \tag{57}$$

3 Examples for the offshell case

3.1 Scalar: $j_L = j_R = 0$

Given k and $j_L = j_R = 0$ we want to find the $GL(4)$ reps K.

Following the prescription in Section 2.3 the $SO(4)$ rep is $\Lambda = [0]$. Taking the inverse of the π projection from equation (148) in Appendix Section A.1 we get

$$\pi^{-1}([0]) = \{\Lambda'\} = \{[0], [1^4]\} \tag{58}$$

Next we add the contractions to get a 4-row K with k boxes.

For $\Lambda' = [0]$, $k' = 0$ and the number of contractions is $t = \frac{k}{2}$. Thus the reps K come from a $GL(4)$ tensor product of $[0]$ with even reps with $k = 2t$ boxes $K = 2T \vdash k$.

$\Lambda' = [1^4]$ corresponds to a single ϵ tensor. $k' = 4$ and the number of contractions is $t = \frac{k-4}{2}$. The reps K come from a $GL(4)$ tensor product of $[1^4]$ with even reps with $k - 4$ boxes $K = [1^4] \circ (2T \vdash k - 4)$.

We find exactly the same expansion of $GL(4)$ reps K by taking the inner product of the two corresponding $GL(2)$ reps:

$$\Lambda_L \otimes \Lambda_R = [\tfrac{k}{2}, \tfrac{k}{2}] \otimes [\tfrac{k}{2}, \tfrac{k}{2}] = \left[[\tfrac{k}{2}](\square\square^{\circ \frac{k}{2}}) + \begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix} \circ [\tfrac{k-4}{2}](\square\square^{\circ \frac{k-4}{2}}) \right]_{\leq 4} = \sum K \tag{59}$$

$[\cdot]_{\leq 4}$ means only keep K if K has 4 or fewer rows, i.e. we are implementing the $GL(4)$ tensor product $\circ_4$. $\otimes$ is the S_k inner product.

***This isn't proved, but true up to $k = 12$. NB: this observation first brought up by Paul in Mathematica file for $k = 6$. Should be able to prove using this paper on the inner product of two-row reps: [1].

This splits into Young diagrams with even and odd row lengths. If we write each diagram in terms of all 4 row lengths, e.g. $[3, 1, 0, 0]$ for $[3, 1]$, then K runs over all Young diagrams of size k with differences between the rows always even ($[3, 1, 0, 0]$ fails this test).

We never need more than one copy of $\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}$ building the Young diagrams because, e.g. $\begin{smallmatrix}\square\square\\\square\square\\\square\square\\\square\square\end{smallmatrix}$ appears in $\square\square\square\square(\square\square^{\circ 4})$.

***Clarify this.

That the LHS of (59) gives the correct offshell counting is proved above.

Following Section 2.5 the operators corresponding to (59) are

$$\begin{aligned}&C^{M'_K}_{M'_{2T}}\; C^{h_1\cdots h_{2t}}_{2T, M'_{2T}}\;\; S^\dagger_{h_1 h_2} \cdots S^\dagger_{h_{2t-1} h_{2t}} |0\rangle \\ &C^{\tilde{\tau}, M'_K}_{M'_{2T}, M'_{[1^4]}}\; \tilde{\Pi}^{[1^4], M_{[1^4]}}_{[0]}\;\; C^{\mu_{2t+1}\cdots\mu_k}_{[1^4], M_{[1^4]}} C^{h_1\cdots h_{2t}}_{2T, M'_{2T}}\; C^{h_{2t+1}\cdots h_k}_{[1^4], M'_{[1^4]}}\;\; S^\dagger_{h_1 h_2} \cdots S^\dagger_{h_{2t-1} h_{2t}} A^\dagger_{h_{2t+1}\mu_{2t+1}} \cdots A^\dagger_{h_k \mu_k} |0\rangle\end{aligned} \tag{60}$$

This covers all independent cases where all indices are contracted and the $SO(4)$ state is trivial. Since $K \in P(k, 4)$ this restricts the number of rows in $2T$.

3.1.1 Explicit examples

For $k = 2$ only $\Lambda' = [0]$ applies and we just get

$$K = \ydiagram{1,1} \otimes \ydiagram{1,1} = \ydiagram{2} \tag{61}$$

For $k = 4$ we also get a contribution from $\Lambda' = [1^4]$ too

$$\begin{aligned} \sum K &= \ydiagram{2,2} \otimes \ydiagram{2,2} = \ydiagram{2}\,(\ydiagram{2}^{\circ 2}) + \ydiagram{1,1,1,1} \\ &= \ydiagram{4} + \ydiagram{2,2} + \ydiagram{1,1,1,1} \end{aligned} \tag{62}$$

For $k = 6$

$$\begin{aligned} \sum K &= \ydiagram{3,3} \otimes \ydiagram{3,3} = \left[\ydiagram{3}\,(\ydiagram{2}^{\circ 3}) + \ydiagram{1,1,1,1} \circ \ydiagram{2}\right]_{\leq 4} \\ &= \ydiagram{6} + \ydiagram{4,2} + \ydiagram{2,2,2} + \ydiagram{3,1,1,1} \\ &= \ydiagram{3}\,(\ydiagram{2}^{\circ 3}) + \ydiagram{1,1,1,1} \circ \ydiagram{2} - \ydiagram{2,1,1,1,1} \end{aligned} \tag{63}$$

In the final line we have rewritten it as a generic $GL(\infty)$ tensor product or general symmetric group outer product, but subtracting the reps with more than 4 rows. This will be useful when we count the onshell operators.

For $k = 8$

$$\begin{aligned} \sum K &= \ydiagram{4,4} \otimes \ydiagram{4,4} = \left[\ydiagram{4}\,(\ydiagram{2}^{\circ 4}) + \ydiagram{1,1,1,1} \circ \ydiagram{2}\,(\ydiagram{2}^{\circ 2})\right]_{\leq 4} \\ &= \ydiagram{8} + \ydiagram{6,2} + \ydiagram{4,4} + \ydiagram{4,2,2} + \ydiagram{2,2,2,2} \\ &\quad + \ydiagram{5,1,1,1} + \ydiagram{3,3,1,1} \\ &= \ydiagram{4}\,(\ydiagram{2}^{\circ 4}) + \ydiagram{1,1,1,1} \circ \ydiagram{2}\,(\ydiagram{2}^{\circ 2}) - \ydiagram{4,1,1,1,1} - \ydiagram{3,2,1,1,1} - \ydiagram{2,2,1,1,1,1} \end{aligned} \tag{64}$$

For $k = 10$

$$\begin{aligned} \sum K &= \ydiagram{5,5} \otimes \ydiagram{5,5} = \left[\ydiagram{5}\,(\ydiagram{2}^{\circ 5}) + \ydiagram{1,1,1,1} \circ \ydiagram{3}\,(\ydiagram{2}^{\circ 3})\right]_{\leq 4} \\ &= \ydiagram{10} + \ydiagram{8,2} + \ydiagram{6,4} + \ydiagram{6,2,2} + \ydiagram{4,4,2} + \ydiagram{4,2,2,2} \\ &\quad + \ydiagram{5,3,1,1} + \ydiagram{7,1,1,1} + \ydiagram{3,3,3,1} \\ &= \ydiagram{5}\,(\ydiagram{2}^{\circ 5}) + \ydiagram{1,1,1,1} \circ \ydiagram{3}\,(\ydiagram{2}^{\circ 3}) - \ydiagram{2,2,2,2,2} \\ &\quad - \ydiagram{6,1,1,1,1} - \ydiagram{5,2,1,1,1} - \ydiagram{4,3,1,1,1} - \ydiagram{4,2,1,1,1,1} - \ydiagram{3,3,2,1,1} - \ydiagram{3,2,2,1,1,1} - \ydiagram{2,2,2,1,1,1,1} \end{aligned} \tag{65}$$

For $k = 12$

For the offshell case

$$\begin{aligned} \sum K &= \ydiagram{6,6} \otimes \ydiagram{6,6} = \left[\ydiagram{6}\,(\ydiagram{2}^{\circ 6}) + \ydiagram{1,1,1,1} \circ \ydiagram{4}\,(\ydiagram{2}^{\circ 4})\right]_{\leq 4} \\ &= \ydiagram{12} + \ydiagram{10,2} + \ydiagram{8,4} + \ydiagram{8,2,2} \\ &\quad + \ydiagram{6,6} + \ydiagram{6,4,2} + \ydiagram{6,2,2,2} + \ydiagram{4,4,4} + \ydiagram{4,4,2,2} \\ &\quad + \ydiagram{9,1,1,1} + \ydiagram{7,3,1,1} + \ydiagram{5,3,3,1} + \ydiagram{5,5,1,1} + \ydiagram{3,3,3,3} \end{aligned} \tag{66}$$

3.2 $j_L - j_R = 0$

Given k and $j_L = j_R = j$ we want to find the $GL(4)$ reps K.

Following the prescription in Section 2.3 the $SO(4)$ rep is $\Lambda = [2j]$. Taking the inverse of the π projection from equation (149) in Appendix Section A.1 we get

$$\pi^{-1}([2j]) = \{\Lambda'\} = \{[2j], [2j, 1, 1], -[2j, 2, 1, 1], -[2j, 2, 2, 2]\} \tag{67}$$

The first two here make sense since

$$\ydiagram{1,1,1} \sim \ydiagram{1} \sim \epsilon^{\mu_1\mu_2\mu_3\mu_4} A^\dagger_{[h_2\mu_2} A^\dagger_{h_3\mu_3} A^\dagger_{h_4]\mu_4} \tag{68}$$

Note that the last two appear for $j \geq 1$ and appear with a minus sign.

Next we add the contractions to get a 4-row K with k boxes.

For $\Lambda' = [2j]$, $k' = 2j$ and the number of contractions is $t = \frac{k-2j}{2}$.

For $\Lambda' = [2j, 1, 1]$, $k' = 2j + 2$ and the number of contractions is $t = \frac{k-2j-2}{2}$.

For $\Lambda' = -[2j, 2, 1, 1]$, $k' = 2j + 4$ and the number of contractions is $t = \frac{k-2j-4}{2}$.

For $\Lambda' = -[2j, 2, 2, 2]$, $k' = 2j + 6$ and the number of contractions is $t = \frac{k-2j-6}{2}$.

3.2.1 $j_L = j_R = \frac{1}{2}$

$k = 1$ is trivial.

For $k = 3$ we get

$$\begin{aligned}\sum K = \Lambda_L \otimes \Lambda_R = \ydiagram{2,1} \otimes \ydiagram{2,1} &= \ydiagram{1} \circ \ydiagram{2} + \ydiagram{1,1,1} \\ &= \ydiagram{3} + \ydiagram{2,1} + \ydiagram{1,1,1}\end{aligned} \tag{69}$$

For $k = 5$ we get

$$\begin{aligned}\sum K = \Lambda_L \otimes \Lambda_R = \ydiagram{3,2} \otimes \ydiagram{3,2} &= \ydiagram{1} \circ \ydiagram{2} (\ydiagram{2}^{\circ 2}) + \ydiagram{1,1,1} \circ \ydiagram{2} \\ &= \ydiagram{5} + \ydiagram{4,1} + \ydiagram{3,2} + \ydiagram{2,2,1} + \ydiagram{3,1,1} + \ydiagram{2,1,1,1}\end{aligned} \tag{70}$$

3.2.2 $j_L = j_R = 1$

$k = 2$ is trivial.

For $k = 4$ we get

$$\begin{aligned}\sum K = \Lambda_L \otimes \Lambda_R = \ydiagram{3,1} \otimes \ydiagram{3,1} &= \ydiagram{2} \circ \ydiagram{2} + \ydiagram{2,1,1} \\ &= \ydiagram{4} + \ydiagram{3,1} + \ydiagram{2,2} + \ydiagram{2,1,1}\end{aligned} \tag{71}$$

For $k = 6$ we get

$$\begin{aligned}\sum K = \Lambda_L \otimes \Lambda_R = \ydiagram{4,2} \otimes \ydiagram{4,2} &= \ydiagram{2} \circ \ydiagram{2} (\ydiagram{2}^{\circ 2}) + \ydiagram{2,1,1} \circ \ydiagram{2} - \ydiagram{2,2,1,1} \\ &= \ydiagram{6} + \ydiagram{5,1} + \ydiagram{4,2} + \ydiagram{4,2} + \ydiagram{3,2,1} + \ydiagram{2,2,2} \\ &\quad + \ydiagram{4,1,1} + \ydiagram{3,2,1} + \ydiagram{3,1,1,1}\end{aligned} \tag{72}$$

This is the first time a Λ' appears with a minus sign; it cancels the appearance of $\ydiagram{2,2,1,1}$ in $\ydiagram{2,1,1} \circ \ydiagram{2}$.

For $k = 8$ we get

$$\sum K = \Lambda_L \otimes \Lambda_R = \ydiagram{5,3} \otimes \ydiagram{5,3} = \ydiagram{2} \circ \ydiagram{3} (\ydiagram{2}^{\circ 3}) + \ydiagram{2,1,1} \circ \ydiagram{2} (\ydiagram{2}^{\circ 2}) - \ydiagram{2,2,1,1} \circ \ydiagram{2} - \ydiagram{2,2,2,2} \tag{73}$$

I've checked this explicitly, but it's too tedious to write out.

3.2.3 $j_L = j_R = \frac{3}{2}$

$k = 3$ is trivial.

For $k = 5$ we get

$$\sum K = \Lambda_L \otimes \Lambda_R = \ydiagram{4,1} \otimes \ydiagram{4,1} = \ydiagram{3} \circ \ydiagram{2} + \ydiagram{3,1,1}$$
$$= \ydiagram{5} + \ydiagram{4,1} + \ydiagram{3,2} + \ydiagram{3,1,1} \tag{74}$$

3.3 $j_L - j_R = 1$

Given k and $j_L = j_R + 1$ we want to find the $GL(4)$ reps K.

Following the prescription in Section 2.3 the $SO(4)$ rep is $\Lambda = [2j_R + 1, 1]$. Taking the inverse of the π projection from equation (150) in Appendix Section A.1 we get

$$\pi^{-1}([2j_R + 1, 1]) = \{\Lambda'\} = \{[2j_R + 1, 1], -[2j_R + 1, 2, 2, 1]\} \tag{75}$$

Next we add the contractions to get a 4-row K with k boxes.

For $\Lambda' = [2j_R + 1, 1]$, $k' = 2j_R + 2$ and the number of contractions is $t = \frac{k-2j_R-2}{2}$.

For $\Lambda' = -[2j_R + 1, 2, 2, 1]$, $k' = 2j_R + 6$ and the number of contractions is $t = \frac{k-2j_R-6}{2}$. This is only a legal diagram for $j_R \geq \frac{1}{2}$.

3.3.1 $j_L = 1, j_R = 0$

$\Lambda = [1, 1]$.

$k = 2$ is trivial.

For $k = 4$ we get

$$\sum K = \Lambda_L \otimes \Lambda_R = \ydiagram{3,1} \otimes \ydiagram{2,2} = \ydiagram{1,1} \circ \ydiagram{2}$$
$$= \ydiagram{3,1} + \ydiagram{2,1,1} \tag{76}$$

For $k = 6$ we get

$$\sum K = \Lambda_L \otimes \Lambda_R = \ydiagram{4,2} \otimes \ydiagram{3,3} = \ydiagram{1,1} \circ \ydiagram{2} (\ydiagram{2}^{\circ 2}) \tag{77}$$

3.3.2 $j_L = \frac{3}{2}, j_R = \frac{1}{2}$

$\Lambda = [2, 1]$.

$k = 3$ is trivial.

For $k = 5$ we get

$$\sum K = \Lambda_L \otimes \Lambda_R = \ydiagram{4,1} \otimes \ydiagram{3,2} = \ydiagram{2,1} \circ \ydiagram{2}$$
$$= \ydiagram{4,1} + \ydiagram{3,2} + \ydiagram{3,1,1} + \ydiagram{2,2,1} \tag{78}$$

For $k = 7$ we get a contribution from $\Lambda' = [2, 2, 2, 1]$ which has to appear with a minus sign to gel with the inner product

$$\sum K = \Lambda_L \otimes \Lambda_R = \ydiagram{5,2} \otimes \ydiagram{4,3} = \ydiagram{2,1} \circ \ydiagram{2} (\ydiagram{2}^{\circ 2}) - \ydiagram{2,2,2,1} \tag{79}$$

This is another example of the important of the minus sign.

3.3.3 $j_L = 2, j_R = 1$

$\Lambda = [3, 1]$.

$k = 4$ is trivial.

For $k=6$ we get

$$\sum K = \Lambda_L \otimes \Lambda_R = \yng(5,1) \otimes \yng(4,2) = \yng(3,1) \circ \yng(2) \\ = \yng(3,1) + \yng(2,1,1) \tag{80}$$

For $k=8$ we get a contribution from $\Lambda' = [3,2,2,1]$ which has to appear with a minus sign to gel with the inner product

$$\sum K = \Lambda_L \otimes \Lambda_R = \yng(6,2) \otimes \yng(5,3) = \yng(3,1) \circ \yng(2) (\yng(2)^{\circ 2}) - \yng(3,2,2,1) \tag{81}$$

3.4 $j_L - j_R = 2$

Following the prescription in Section 2.3 the $SO(4)$ rep is $\Lambda = [2j_R + 2, 2]$. Taking the inverse of the π projection from equation (151) in Appendix Section A.1 we get

$$\pi^{-1}([2j_R + 2, 2]) = \{\Lambda'\} = \{[2j_R + 2, 2], -[2j_R + 2, 2, 2]\} \tag{82}$$

3.4.1 $j_L = 2, j_R = 0$

For $k=4$ this is trivial

$$\sum K = \yng(4) \otimes \yng(2,2) = \yng(2,2) \tag{83}$$

For $k=6$ we get a non-trivial contribution with a minus sign

$$\sum K = \yng(5,1) \otimes \yng(3,3) = \yng(2,2) \circ \yng(2) - \yng(2,2,2) \\ = \yng(4,2) + \yng(3,2,1) \tag{84}$$

3.5 $j_L - j_R = 3$

Following the prescription in Section 2.3 the $SO(4)$ rep is $\Lambda = [2j_R + 3, 3]$. Taking the inverse of the π projection from equation (152) in Appendix Section A.1 we get

$$\pi^{-1}([2j_R + 3, 3]) = \{\Lambda'\} = \{[2j_R + 3, 3], -[2j_R + 3, 3, 2], [2j_R + 3, 3, 3, 1], -[2j_R + 3, 3, 3, 3]\} \tag{85}$$

3.5.1 $j_L = 3, j_R = 0$

For $k=6$ this is trivial

$$\sum K = \yng(6) \otimes \yng(3,3) = \yng(3,3) \tag{86}$$

For $k=8$

$$\sum K = \yng(7,1) \otimes \yng(4,4) = \yng(3,3) \circ \yng(2) - \yng(3,3,2) \\ = \yng(6,2) + \yng(4,3,1) \tag{87}$$

For $k=10$

$$\sum K = \yng(8,2) \otimes \yng(5,5) = \yng(3,3) \circ \yng(2) (\yng(2)^{\circ 2}) - \yng(3,3,2) \circ \yng(2) + \yng(3,3,3,1) \\ = \yng(7,3) + \yng(6,3,1) + \yng(5,5) + \yng(5,4,1) + \yng(5,3,2) + \yng(4,4,1,1) \tag{88}$$

For $k=12$

$$\sum K = \yng(9,3) \otimes \yng(6,6) = \yng(3,3) \circ \yng(3) (\yng(2)^{\circ 3}) - \yng(3,3,2) \circ \yng(2) (\yng(2)^{\circ 2}) + \yng(3,3,3,1) \circ \yng(2) - \yng(3,3,3,3) \tag{89}$$

4 The onshell case

4.1 The onshell operator

We want to remove the equations of motion for individual fields $\partial^\mu \partial_\mu X = 0$, i.e. when two $a^\dagger_{i\mu}$ act on the same place labelled by i and have their $SO(4)$ indices contracted by η

$$\eta^{\mu_1\mu_2} a^\dagger_{i\mu_1} a^\dagger_{i\mu_2} \tag{90}$$

There is no summation over i. It is clear that we must work in the $SO(4)$ formalism to do this.

For our HWS consider the contraction of two hooks V_H

$$\eta^{\mu_1\mu_2} A^\dagger_{h_1\mu_1} A^\dagger_{h_2\mu_2} \tag{91}$$

Because η is symmetric, as a representation of S_n this transforms in $\Box\Box(V_H^{\circ 2}) = V_{\rm nat} \oplus V_{[n-2,2]}$. To apply the EoM we just remove the diagonal $V_{\rm nat}$ (which corresponds to when $\partial^\mu\partial_\mu$ are acting on the same site) from $\Box\Box(V_H^{\circ 2})$ to get $V_B \equiv V_{[n-2,2]}$. Thus whenever we contract two hooks, we must project to V_B

$$\eta^{\mu_1\mu_2} A^\dagger_{h_1\mu_1} A^\dagger_{h_2\mu_2} \to B^\dagger_{h_1h_2} \equiv P^{h'_1h'_2}_{h_1h_2} S^\dagger_{h'_1h'_2} = P^{h'_1h'_2}_{h_1h_2} \eta^{\mu_1\mu_2} A^\dagger_{h'_1\mu_1} A^\dagger_{h'_2\mu_2} \tag{92}$$

There is more detail on this projection in note-EOM. If we feed this projected contraction into the offshell operator (32) we find

$$\begin{aligned} &\left|K, \tilde{M}'_K, 2T, \Lambda', \Lambda, M_\Lambda, \tilde{\tau}\right\rangle \\ &= \sum_{a_{\Lambda'}} C^{\tilde{\tau},\tilde{M}'_K}_{\tilde{M}'_{2T},M'_{\Lambda'}} \tilde{\Pi}^{\Lambda',M_{\Lambda'}}_{\Lambda,M_\Lambda} C^{\mu_{2t+1}\cdots\mu_k}_{\Lambda',M_{\Lambda'},a_{\Lambda'}} C^{h_1\cdots h_{2t}}_{2T,\tilde{M}'_{2T}} C^{h_{2t+1}\cdots h_k}_{\Lambda',M'_{\Lambda'},a_{\Lambda'}} \; B^\dagger_{h_1h_2}\cdots B^\dagger_{h_{2t-1}h_{2t}} A^\dagger_{h_{2t+1}\mu_{2t+1}} \cdots A^\dagger_{h_k\mu_k}|0\rangle \end{aligned} \tag{93}$$

It's important to note that we've had to modify the $GL(d_H)$ state to $\tilde{M}'_{2T}$ of $2T$ to account for the fact that we've projected out the equation of motion terms. It's not clear that this really corresponds to a $GL(d_H)$ rep anymore.

A $GL(d_H)$ description of $Sym(V_B^{\otimes t})$ is useful to get one description of the counting but not essential. As explained in more detail in symvb.tex we have

$$Sym(V_B^{\otimes t}) = \bigoplus_{2T,\lambda_2} V_{2T,\phi} \otimes V_{\lambda_2} \otimes V_{2T,\lambda_2} \tag{94}$$

$V_{2T,\phi}$ is a 1-dimensional space corresponding to the even rep. $2T$ of S_{2t} which transforms as the trivial of the $S_2^t \ltimes S_t$ subgroup. The existence of the decomposition 94 is also useful for replacing $C^{h_1\cdots h_{2t}}_{2T,\tilde{M}'_{2T}}$ which manifestly makes sense. We replace it with $C^{h_1\cdots h_{2t}}_{2T,\lambda_2,a_{\lambda_2},\tau_{2T,\lambda_2}}$. The state τ_{2T,λ_2} runs over $dimV_{2T,\lambda_2}$. We can also decompose the $GL(d_H)$ state $M'_{\Lambda'}$ into S_n states :

$$V^{(GL(d_H))}_{\Lambda'} = \bigoplus_{\lambda_3} V^{\lambda_3}_{(S_n)} \otimes V^{\lambda_3}_{\Lambda'} \tag{95}$$

with a multiplicity label $\tau_{\Lambda',\lambda_3}$ running over $DimV^{\lambda_3}_{\Lambda'}$. So we will have the corresponding Clebsch $C^{\Lambda',M'_{\Lambda'}}_{\lambda_3,a_{\lambda_3},\tau_{\Lambda',\lambda_3}}$. We can couple the resulting S_n state a_{λ_3} with the state a_{λ_2} with an S_n inner Clebsch $C^{\lambda_3,\lambda_2,a_{\lambda_2},a_{\lambda_3}}_{4(2T,\Lambda');\lambda,a_\lambda}$ constrained by the $GL(4)$ cutoff. The subscript $4(2T,\Lambda')$ indicates that S_n reps coming from Λ' and S_n reps which were coupled to $2T$ are coupled to only the λ which are constrained to by the requirement that $2T \otimes \Lambda'$ does not have more than 4 rows.

The formula gets longer, but teh steps are simple :

$$\begin{aligned} &|\lambda(S_n), \lambda_2(S_n), \lambda_3(S_n), \tau_{2T,\lambda_2}, \tau_{\Lambda',\lambda_2}, 2T(S_{2t}), \Lambda', \Lambda(so(4)), M_\Lambda\rangle \\ &= C^{\lambda_3,\lambda_2,a_{\lambda_2},a_{\lambda_3}}_{4(2T,\Lambda');\lambda,a_\lambda} C^{\Lambda',M'_{\Lambda'}}_{\lambda_3,a_{\lambda_3},\tau_{\Lambda',\lambda_3}} \tilde{\Pi}^{\Lambda',M_{\Lambda'}}_{\Lambda,M_\Lambda} C^{h_1\cdots h_{2t}}_{2T,\lambda_2,a_{\lambda_2},\tau_{2T,\lambda_2}} C^{\mu_{2t+1}\cdots\mu_k}_{\Lambda',M_{\Lambda'},a_{\Lambda'}} C^{h_{2t+1}\cdots h_k}_{\Lambda',M'_{\Lambda'},a_{\Lambda'}} \\ &B^\dagger_{h_1h_2}\cdots B^\dagger_{h_{2t-1}h_{2t}} A^\dagger_{h_{2t+1}\mu_{2t+1}} \cdots A^\dagger_{h_k\mu_k}|0\rangle \end{aligned} \tag{96}$$

In the above ket, we have made explicit what group the rep. label belongs to, so the formula is easier to read. The label same Λ' is used for $GL(4), GL(d_H), S_{k'}$, which is due to Schur-Weyl duality and $S_{k'}$ symmetry, which is explained in more detail in section C.2. All repeated state-label indices are summed.

Where do the $GL(4)$ cutoffs appear in the above formula ? Λ' is a $GL(4)$ label, also shared by other groups, so it imposes $GL(4)$ cutoffs. Compared to 93 the updated 96 has lost the $GL(d_H)$-Clebsch $C^{\tilde{\tau},\tilde{M}'_K}_{\tilde{M}'_{2T},M'_{\Lambda'}}$. When we had $GL(d_H)$ states in the off-shell case, it was easy to state how the $GL(4)$ cutoff acts. It is clear we still need a $GL(4)$ cutoff, but now it has to constrain the S_n tensor product. (the need for this cutoff can be seen in the $k = 6$ example).

Exercise 1 : Can we try and write a formula for $C^{\lambda_3,\lambda_2,a_{\lambda_2},a_{\lambda_3}}_{4;\lambda,a_\lambda}$ which makes a little clearer how the 4-cutoff operates. The words describing it above are probably enough to see how it works in examples such as the $k = 6$ below, but a neat general formula would be good.

Exercise 2 : That should allow us to write a counting formula which is built as a sum of products of manifestly positive multiplicities, but equal to the alternating sum formulae.

4.2 $SO(4)$ onshell counting

To account for the loss of these terms in the counting we need to proceed carefully.

In the offshell case for the η contractions we had

$$\mathrm{Sym}(\mathrm{Sym}(V_H^{\circ 2})^{\circ t}) = [t](\yng(2)(V_H^{\circ 2})^{\circ t}) = \sum_{2T\in P(2t)} 2T(V_H^{\circ 2t}) \tag{97}$$

By expanding into even Young diagrams with V_H indices we could easily see the $GL(4)$ contraint. Here we have

$$\mathrm{Sym}(V_B^{\circ t}) \tag{98}$$

How do we translate into V_H indices so we can see the $GL(4)$ contraints?

Comment : The dual of $GL(4)$ is S_k on $W^{\otimes k}$. On $W_1^{\otimes 2t} \otimes W_2^{\otimes k'}$ it is $S_{2t} \times S_{k'}$. As we see in sections C.2 and C.3, we expect the $GL(4)$ cutoff to always be expressed in terms of its duals. A $GL(d_H)$ presentation is possible as follows, but should not be essential.

The answer is to write $V_B = \yng(2)(V_H^{\circ 2}) - V_{\mathrm{nat}}$ and perform the alternating expansion

$$\begin{aligned}
[t](V_B^{\circ t}) &= [t]\left((\yng(2)(V_H^{\circ 2}) - V_{\mathrm{nat}})^{\circ t}\right) \\
&= \sum_{p=0}^{t} (-1)^p [1^p](V_{\mathrm{nat}}^{\circ p}) \circ [t-p]\left((\yng(2)(V_H^{\circ 2}))^{\circ t-p}\right) \\
&= \sum_{p=0}^{t} \sum_{2T\in P(2t-2p)} (-1)^p [1^p](V_{\mathrm{nat}}^{\circ p}) \circ 2T(V_H^{\circ 2t-2p})
\end{aligned} \tag{99}$$

For example $t = 3$

$$\begin{aligned}
\yng(3)(V_B^{\circ 3}) &= \yng(3)((\yng(2)(V_H^{\circ 2}))^{\circ 3}) - V_{\mathrm{nat}} \circ \yng(2)((\yng(2)(V_H^{\circ 2}))^{\circ 2}) + \yng(1,1)(V_{\mathrm{nat}}^{\circ 2}) \circ \yng(2)(V_H^{\circ 2}) - \yng(1,1,1)(V_{\mathrm{nat}}^{\circ 3}) \\
&= \left(\yng(6) + \yng(4,2) + \yng(2,2,2)\right)(V_H^{\circ 6}) - V_{\mathrm{nat}} \circ \left(\yng(4) + \yng(2,2)\right)(V_H^{\circ 4}) + \yng(1,1)(V_{\mathrm{nat}}^{\circ 2}) \circ \yng(2)(V_H^{\circ 2}) - \yng(1,1,1)(V_{\mathrm{nat}}^{\circ 3})
\end{aligned} \tag{100}$$

So to apply the $GL(4)$ constraint properly here, whenever we tensor $2T(V_H^{\circ 2t-2p})$ with $\Lambda'(V_H^{\circ k'})$ we must restrict the result $K \vdash k - 2p$ to 4 rows.

For a given $SO(4)$ rep Λ and dimension $\Delta = n + k$ the counting inherits the alternating sum (cf. the offshell formula (38))

$$\mathrm{mult}_{\mathrm{EoM}}(\Lambda, \Delta) = \sum_{p=0}^{t} (-1)^p \sum_{K\in P(k-2p,4)} \sum_{k'} \sum_{\Lambda'\vdash k',2T\vdash k-k'-2p} \tilde{g}(2T, \Lambda'; K)\, \delta(\Lambda = \tilde{\pi}(\Lambda'))\ \mathrm{Dim}_{d_{\mathrm{nat}}}[1^p]\ \mathrm{Dim}_{d_H} K \tag{101}$$

Refining to a specific S_n rep λ (cf. offshell version (39)) we must expand out

$$[1^p](V_{\mathrm{nat}}^{\circ p}) \otimes K(V_H^{\circ k-2p}) \tag{102}$$

into S_n reps. This is done in detail below.

We prove these formulae below using the $SU(2)_L \times SU(2)_R$ character expansion.

4.3 Onshell character expansion

For the character of V_F we must now apply the EoM and remove terms like $\partial_\mu \partial^\mu X$ from V_F. This gives a character

$$\chi_F = \chi_{1,0,0} = P(1-s^2)s \tag{103}$$

For $n \geq 3$ the characters are not modified from the off-shell case

$$\chi_{\Delta,j_L,j_R} = P s^\Delta \chi_{j_L}(X)\chi_{j_R}(Y) \tag{104}$$

***What was the story with $n = 2$?

Expanding the character for $V_F^{\otimes n}$

$$\begin{aligned}
\chi_F^n &= \left[P(1-s^2)s\right]^n \\
&= P(1-s^2)^n s^n \sum_{q=0}^{\infty} s^q \sum_{\Lambda_L,\Lambda_R,\Lambda_2 \vdash q} \sum_{\lambda_1 \vdash n} d_{\lambda_1} \, \mathrm{mult}(V_H^{\otimes q}, \lambda_1 \otimes \Lambda_2) C(\Lambda_L,\Lambda_R,\Lambda_2)\chi_{\Lambda_L}(X)\chi_{\Lambda_R}(Y) \\
&= P s^n \sum_{p=0}^{n} (-1)^p s^{2p} \binom{n}{p} \sum_{q=0}^{\infty} s^q \sum_{\Lambda_L,\Lambda_R,\Lambda_2 \vdash q} \sum_{\lambda_1 \vdash n} d_{\lambda_1} \, \mathrm{mult}(V_H^{\otimes q}, \lambda_1 \otimes \Lambda_2) C(\Lambda_L,\Lambda_R,\Lambda_2)\chi_{\Lambda_L}(X)\chi_{\Lambda_R}(Y)
\end{aligned} \tag{105}$$

Now make the crucial step of identifying the binomial coefficient with the antisymmetric product of V_{nat}'s that appears in the expansion of V_B in (99)

$$\binom{n}{p} = \dim\,[1^p](V_{\mathrm{nat}}^{\circ p}) \tag{106}$$

Collect powers of s^k where $k = 2p + q$

$$P s^n \sum_{k=0}^{\infty} s^k \sum_{p=0}^{n} (-1)^p d_{[\mathrm{anti\,nat}^{\otimes p}]} \sum_{\Lambda_L,\Lambda_R,\Lambda_2 \vdash k-2p} \sum_{\lambda_1 \vdash n} d_{\lambda_1} \, \mathrm{mult}(V_H^{\otimes k-2p}, \lambda_1 \otimes \Lambda_2) C(\Lambda_L,\Lambda_R,\Lambda_2)\chi_{\Lambda_L}(X)\chi_{\Lambda_R}(Y) \tag{107}$$

Obviously the summand vanishes if $k - 2p < 0$. We see that each time we increase p we drop the number of boxes available for the $\Lambda_L \otimes \Lambda_R$ inner product by two and increase the number of anti-symmetrised V_{nat} by one.

Next take the tensor product $V_{[\mathrm{anti\,nat}^{\otimes p}]} \otimes V_{\lambda_1}$

$$d_{[\mathrm{anti\,nat}^{\otimes p}]} d_{\lambda_1} = \sum_{\lambda \vdash n} C([\mathrm{anti\,nat}^{\otimes p}], \lambda_1, \lambda) d_\lambda \tag{108}$$

and rearrange

$$\begin{aligned}
\chi_F^n = & P s^n \sum_{k,j_L,j_R=0}^{\infty} s^k \chi_{j_L}(X)\chi_{j_R}(Y) \sum_{\lambda \vdash n} d_\lambda \\
& \sum_{p=0}^{n} (-1)^p \sum_{\lambda_1 \vdash n} C([\mathrm{anti\,nat}^{\otimes p}], \lambda_1, \lambda) \sum_{\Lambda_2 \vdash k-2p} \mathrm{mult}(V_H^{\otimes k-2p}, \lambda_1 \otimes \Lambda_2) \\
& C\left(\Lambda_L = \{k-2p, j_L\}, \Lambda_R = \{k-2p, j_R\}, \Lambda_2\right)
\end{aligned} \tag{109}$$

What is really going on here? We take the original V_λ with EoM and for each p we are removing some of the λ, via anti $\left(V_{\mathrm{nat}}^{\otimes p}\right) = V_{[n-p+1,1^{p-1}]} \oplus V_{[n-p,1^p]}$.

This result matches with our goal (49)

$$\begin{aligned}
\mathrm{mult}_{\mathrm{EoM}}(\Delta = n+k, j_L, j_R, \lambda) = & \sum_{p=0}^{n} (-1)^p \sum_{\lambda_1 \vdash n} C([\mathrm{anti\,nat}^{\otimes p}], \lambda_1, \lambda) \sum_{\Lambda_2 \vdash k-2p} \mathrm{mult}(V_H^{\otimes k-2p}, \lambda_1 \otimes \Lambda_2) \\
& C\left(\Lambda_L = \{k-2p, j_L\}, \Lambda_R = \{k-2p, j_R\}, \Lambda_2\right)
\end{aligned} \tag{110}$$

More readably we could write this

$$\mathrm{mult}_{\mathrm{EoM}}(\Delta, j_L, j_R, \lambda) = \text{ number of times } \lambda \text{ appears in } \sum_{p=0}^{n} (-1)^p \left\{ [1^p](V_{\mathrm{nat}}^{\circ p}) \circ [\Lambda_L \otimes \Lambda_R] \, (V_H^{\otimes k-2p}) \right\} \tag{111}$$

Each time we increase p we remove a column from each of Λ_L and Λ_R.

If we're not interested in the S_n multiplicity then

$$\begin{aligned}\mathrm{mult}_{\mathrm{EoM}}(\Delta, j_L, j_R) &= \sum_{\lambda} d_\lambda \ \mathrm{mult}_{\mathrm{EoM}}(\Delta, j_L, j_R, \lambda) \\ &= \sum_{p=0}^{n} \sum_{K \vdash k-2p} (-1)^p \ C(\Lambda_L \otimes \Lambda_R, K) \ \mathrm{Dim}_{d_{\mathrm{nat}}}[1^p] \ \mathrm{Dim}_{d_H} K \end{aligned} \tag{112}$$

This matches (101).

5 Examples for the onshell case

5.1 Scalar: $j_L = j_R = 0$

Compare this section to its offshell equivalent in Section 3.1. In the decomposition of $GL(4)$ reps K in (59) we now just substitute $\ydiagram{2}(V_H^{\circ 2})$ with V_B. However we must be aware of the alternating expansion of $[t](V_B)$ when we enforce the $GL(4)$ tensor products. If we now do the expansion of S_n reps we get

$$\begin{aligned} &\left[[\tfrac{k}{2}](V_B^{\circ \frac{k}{2}}) + \ydiagram{1,1,1,1}(V_H^{\circ 4}) \circ [\tfrac{k-4}{2}](V_B^{\circ \frac{k-4}{2}}) \right]_{\leq 4} \\ &= [\tfrac{k}{2}](V_B^{\circ \frac{k}{2}}) + \ydiagram{1,1,1,1}(V_H^{\circ 4}) \circ [\tfrac{k-4}{2}](V_B^{\circ \frac{k-4}{2}}) \\ &\quad - \sum_{p=0}^{\frac{k-6}{2}} (-1)^p [1^p](V_{\mathrm{nat}}^{\circ p}) \circ \left[[\tfrac{k-2p}{2}](\ydiagram{2}(V_H^{\circ 2})^{\circ \frac{k-2p}{2}}) + \ydiagram{1,1,1,1}(V_H^{\circ 4}) \circ [\tfrac{k-2p-4}{2}](\ydiagram{2}(V_H^{\circ 2})^{\circ \frac{k-2p-4}{2}}) \right]_{>4} \end{aligned} \tag{113}$$

In the second line, just as we have done in the explicit offshell examples in Section 3.1.1, we have written the $GL(4)$ tensor product first as an unconstrained $GL(\infty)$ tensor product followed by the subtraction of reps with more than 4 rows.

In terms of operators the counting in (113) corresponds to the operators

$$\begin{aligned} &B^\dagger_{h_1 h_2} \cdots B^\dagger_{h_{2t-1} h_{2t}} |0\rangle \\ &B^\dagger_{h_1 h_2} \cdots B^\dagger_{h_{2t-1} h_{2t}} \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} A^\dagger_{[h_1 \mu_1} \cdots A^\dagger_{h_4] \mu_4} |0\rangle \end{aligned} \tag{114}$$

This covers all independent cases where all indices are contracted.

Not that for these operators Young diagrams with more than four rows just vanish because there are only 4 μ indices.

In this section we get sloppy with notation and write $\ydiagram{2}(V_H)$ instead of $\ydiagram{2}(V_H^{\circ 2})$ - it should be obvious from the number of boxes what space we're symmetrising.

5.1.1 $k = 2, j_L, j_R = 0$

For the offshell case we have

$$\left(\ydiagram{1,1} \otimes \ydiagram{1,1} \right)(V_H) = \ydiagram{2}(V_H) \tag{115}$$

For the onshell case we have

$$V_B \tag{116}$$

corresponding to the operator

$$B^\dagger_{h_1 h_2} |0\rangle \tag{117}$$

5.1.2 $k = 4, j_L, j_R = 0$

For the offshell case we have

$$\left(\ydiagram{2,2} \otimes \ydiagram{2,2} \right)(V_H) = \ydiagram{2}\left(\ydiagram{2}(V_H)\right) + \ydiagram{1,1,1,1}(V_H) = \ydiagram{4}(V_H) + \ydiagram{2,2}(V_H) + \ydiagram{1,1,1,1}(V_H) \tag{118}$$

For the onshell case we have

$$\yng(2)\,(V_B) + \yng(1,1,1,1)(V_H) \tag{119}$$

corresponding to the operators

$$B^\dagger_{h_1 h_2} \cdots B^\dagger_{h_3 h_4}|0\rangle \quad \text{and} \quad \epsilon^{\mu_1\mu_2\mu_3\mu_4} A^\dagger_{[h_1\mu_1} \cdots A^\dagger_{h_4]\mu_4}|0\rangle \tag{120}$$

This correctly gives the number of HWS with these quantum numbers and EoM

$$\frac{(n-1)^2(n-2)(n-3)}{6} \tag{121}$$

5.1.3 $k = 6, j_L, j_R = 0$

Here is where problems originally occured in Paul's Mathematica file. That problem turned out to be generic.

For the offshell case we have

$$\begin{aligned}\left(\yng(3,3) \otimes \yng(3,3)\right)(V_H) &= \left[\yng(3)\,(\yng(2)\,(V_H)) + \yng(1,1,1,1)(V_H) \circ \yng(2)\,(V_H)\right]_{\leq 4} \\ &= \yng(6)\,(V_H) + \yng(4,2)\,(V_H) + \yng(2,2,2)\,(V_H) + \yng(3,1,1,1)\,(V_H)\end{aligned} \tag{122}$$

For the onshell case we have

$$\left[\yng(3)\,(V_B) + \yng(1,1,1,1)(V_H) \circ V_B\right]_{\leq 4} = \yng(3)\,(V_B) + \yng(1,1,1,1)(V_H) \circ V_B - \yng(2,1,1,1,1)\,(V_H) \tag{123}$$

corresponding to the operators

$$B^\dagger_{h_1 h_2} B^\dagger_{h_3 h_4} B^\dagger_{h_5 h_6}|0\rangle \quad \text{and} \quad B^\dagger_{h_1 h_2} \epsilon^{\mu_3\mu_4\mu_5\mu_6} A^\dagger_{[h_3\mu_3} \cdots A^\dagger_{h_6]\mu_6}|0\rangle \tag{124}$$

This correctly gives the number of HWS with these quantum numbers and EoM

$$\frac{n(n-1)(n-2)(n-3)(5n^2 - 21n + 28)}{144} \tag{125}$$

This example shows the need for the $C^{\cdots}_{4(2T,\Lambda'),\ldots}$, i.e the $GL(4)$ corrected S_n Clebschs. A simple example fo exercise (1) is to do in it this case.

5.1.4 $k = 8, j_L, j_R = 0$

From an $SO(4)$ point of view, this can happen in two different ways

$$\eta\eta\eta\eta \tag{126}$$

$$\eta\eta\epsilon \tag{127}$$

One might think that

$$\epsilon\epsilon \tag{128}$$

is a separate case, but it is one of (126) when they're antisymmetrised.

***Clarify this.

For the offshell case we have

$$\begin{aligned}\yng(4,4) \otimes \yng(4,4) =& \yng(8) + \yng(6,2) + \yng(4,4) + \yng(4,2,2) + \yng(2,2,2,2) \\ &+ \yng(5,1,1,1) + \yng(3,3,1,1)\end{aligned} \tag{129}$$

The first 5 cases are (126); the last 2 are (127).

We write first 5 cases as

$$\yng(4)\,(\yng(2)) \tag{130}$$

The last two cases are roughly

$$\ydiagram{1,1,1} \circ \ydiagram{2}(\ydiagram{2}) \tag{131}$$

But we must remember that we only allow $GL(4)$ reps so we must remove extra stuff since

$$\ydiagram{1,1,1} \circ \ydiagram{2}(\ydiagram{2}) = \ydiagram{1,1,1} \circ \left(\ydiagram{4} + \ydiagram{2,2}\right) = \ydiagram{5,1,1} + \ydiagram{4,1,1,1} + \ydiagram{3,3,1} + \ydiagram{3,2,1,1} + \ydiagram{2,2,1,1,1} \tag{132}$$

Thus

$$\Lambda_L \otimes \Lambda_R = \ydiagram{3,3} \otimes \ydiagram{3,3} = \ydiagram{4}(\ydiagram{2}) + \ydiagram{1,1,1} \circ \ydiagram{2}(\ydiagram{2}) - \ydiagram{4,1,1,1} - \ydiagram{3,2,1,1} - \ydiagram{2,2,1,1,1} \tag{133}$$

For the onshell case we substitute $\ydiagram{2}$ with V_B.

$$\begin{aligned}
&\left[\ydiagram{4}(V_B) + \ydiagram{1,1,1}(V_H) \circ \ydiagram{2}(V_B)\right]_{\leq 4} \\
&= \ydiagram{4}(V_B) + \ydiagram{1,1,1}(V_H) \circ \ydiagram{2}(V_B) - \ydiagram{4,1,1,1}(V_H) - \ydiagram{3,2,1,1}(V_H) - \ydiagram{2,2,1,1,1}(V_H) + V_{\text{nat}} \circ \ydiagram{2,1,1,1}(V_H)
\end{aligned} \tag{134}$$

This correctly gives the number of HWS with these quantum numbers and EoM

$$\frac{n(n-1)(n-2)(n-3)(7n^4 - 48n^3 + 143n^2 - 222n + 180)}{1440} \tag{135}$$

5.1.5 $k = 10, j_L, j_R = 0$

For the offshell case

$$\begin{aligned}
\ydiagram{5,5} \otimes \ydiagram{5,5} =& \ydiagram{10} + \ydiagram{8,2} + \ydiagram{6,4} + \ydiagram{6,2,2} + \ydiagram{4,4,2} + \ydiagram{4,2,2,2} \\
&+ \ydiagram{5,3,1,1} + \ydiagram{7,1,1,1} + \ydiagram{3,3,3,1}
\end{aligned} \tag{136}$$

For the onshell case

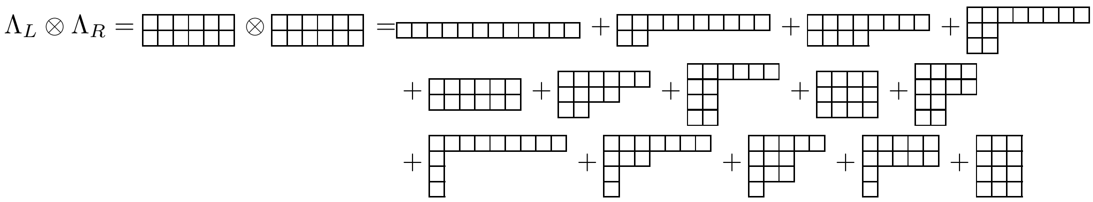
$$\begin{aligned}
&\left[\ydiagram{5}(V_B) + \ydiagram{1,1,1}(V_H) \circ \ydiagram{3}(V_B)\right]_{\leq 4} \\
=& \ydiagram{5}(V_B) + \ydiagram{1,1,1,1}(V_H) \circ \ydiagram{3}(V_B) \\
&- \ydiagram{2,2,2,2,2}(V_H) \\
&- \ydiagram{6,1,1,1,1}(V_H) - \ydiagram{5,2,1,1,1}(V_H) - \ydiagram{4,3,1,1,1}(V_H) - \ydiagram{4,2,1,1,1,1}(V_H) - \ydiagram{3,3,2,1,1}(V_H) - \ydiagram{3,2,2,1,1,1}(V_H) - \ydiagram{2,2,2,1,1,1,1}(V_H) \\
&+ V_{\text{nat}} \circ \left(\ydiagram{4,1,1,1,1}(V_H) + \ydiagram{3,2,1,1,1}(V_H) + \ydiagram{2,2,1,1,1,1}(V_H)\right) \\
&- \ydiagram{1,1}(V_{\text{nat}}) \circ \ydiagram{2,1,1,1,1}(V_H)
\end{aligned} \tag{137}$$

This correctly gives the number of HWS with these quantum numbers and EoM

$$\frac{n(n-1)(n-2)(n-3)(7n^6 - 63n^5 + 285n^4 - 825n^3 + 1608n^2 - 276n + 1280)}{14400} \tag{138}$$

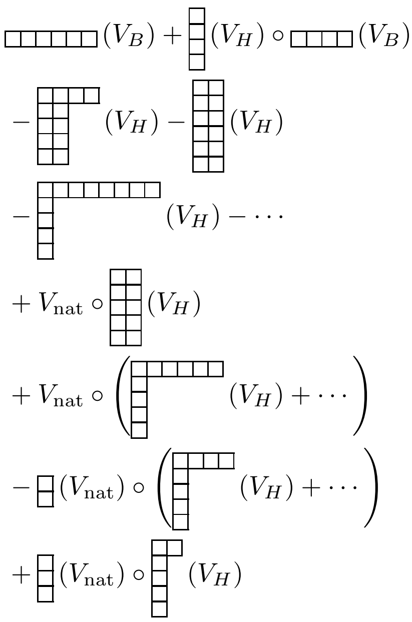
5.1.6 $k = 12, j_L, j_R = 0$

For the offshell case



(139)

For the onsheel case the correct way of getting this is detailed in a SAGE file.



(140)

The general formula is below.

It correctly gives dimension

$$\frac{n(n-1)(n-2)(n-3)(11n^8 - 117n^7 + 702n^6 - 2960n^5 + 9219n^4 - 21083n^3 + 34588n^2 - 36320n + 21000)}{302400} \tag{141}$$

6 An incorrect theorem

One can expand V_B and V_{nat} in terms of V_H. One might think one could just then expand

$$[\tfrac{k}{2}](V_B) + [1^4](V_H) \circ [\tfrac{k}{2} - 2](V_B) \tag{142}$$

in terms of V_H and then throw away reps with more than 4 rows. This doesn't work, see A4 notebook 23/3/09. One needs to ignore the $[1^p](V_{\text{nat}})$ when throwing away rows, as in (113). I don't understand why.

A π projection

In this section Young diagrams are mostly written in terms of their columns lengths, i.e. we write $[2^{k_2}, 1^{k_1}]^T$ instead of $[k_1 + k_2, k_2]$.

We follow the decomposition in Koike and Terada [2].

To decompose a representation K of $GL(2n)$ into representations Λ of $SO(2n)$ we first remove all possible combinations of contractions η from K to get a Young diagram Λ'. Then we project it to an n-row representation Λ of $SO(2n)$ with π.

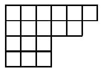
$$K = \bigoplus_{2T,\Lambda'} g(2T, \Lambda'; K) \;\; \pi(\Lambda') = \bigoplus_{\Lambda} \dim V_{K,\Lambda} \;\; \Lambda \tag{143}$$

We have summed over even partitions $2T$ which correspond to contractions η.

The projection π works as follows

- List the l column lengths of Λ'.
- Fold the columns up at $n+i-1$, where $i \in \{1, \dots l\}$ labels each column. Define $\vec{k}$ after cancelling folded with unfolded boxes. For $SO(4)$, i.e. $n = 2$, this means that if the first column is of length 4, replace it with one of length $k_1 = 0$; if the first column is 3, replace it by $k_1 = 1$; if the second column is 4 replace it by $k_2 = 2$.
- Define $\vec{t}$ by $t_i = k_i - i + 1$.
- Define $\vec{T}$ by re-ordering $\vec{t}$ so that $T_j = t_{\sigma(i)}$ for some permutation $\sigma \in S_l$ and $n \geq T_1 > T_2 > \cdots > T_l$.
- Define $\vec{\mu}$ by $\mu_i = T_i + i - 1$. These are the column lengths of $\Lambda = \pi(\Lambda')$.
- It appears with a sign given by the sign of the permutation σ.

As an example take $\Lambda' = [6, 5, 3, 3] = [4, 4, 4, 2, 2, 1]^T$ for $n = 2$ and project it to Λ of $SO(4)$.

$\Lambda' =$  (144)

Folding up we get $\vec{k} = (0, 2, 4, 2, 2, 1)$. Applying the subtraction we get $\vec{t} = (0, 1, 2, -1, -2, -4)$. Rearranging by size we get $T = (2, 1, 0, -1, -2, -4)$ and $\sigma = (13)$. Finally doing the addition $\Lambda = -[2, 2, 2, 2, 2, 1]^T$ where the sign is the sign of the permutation $\sigma = (13)$.

Diagrams with two rows left the same

$$\pi([2^{k_2}, 1^{k_1}]^T) = [2^{k_2}, 1^{k_1}]^T \tag{145}$$

for $k_1, k_2 \geq 0$.

For diagrams with three rows

$$\begin{aligned} \pi([3, 1^k]^T) &= [1^{k+1}]^T \\ \pi([3, 2, *]^T) &= 0 \\ \pi([3, 3, 2^{k_2}, 1^{k_1}]^T) &= -[2^{k_2+2}, 1^{k_1}]^T \\ \pi([3, 3, 3, *]^T) &= 0 \end{aligned} \tag{146}$$

for $k, k_1, k_2 \geq 0$. $*$ represents any column lengths that give a legitimate Young diagram.

The first line is pretty intuitive. A column of length 3 along with k columns of length 1 is replaced by a new Young diagram where we have $k+1$ columns of length 1. Equivalently the projected Young diagram has a row of length $[k+1]$. Note the sign in the third line.

For diagrams with four rows the non-zero projections are

$$\begin{aligned} \pi([4]^T) &= [0]^T = 1 \text{ dim. rep.} \\ \pi([4, 2, 1^k]^T) &= -[1^{k+2}]^T \\ \pi([4, 3, 1^k]^T) &= -[2, 1^{k+1}]^T \\ \pi([4, 3, 3, 2^{k_2}, 1^{k_1}]^T) &= [2^{k_2+3}, 1^{k_1}]^T \\ \pi([4, 4, 1^k]^T) &= -[1^{k+2}]^T \\ \pi([4, 4, 4, 2^{k_2}, 1^{k_1}]^T) &= -[2^{k_2+3}, 1^{k_1}]^T \end{aligned} \tag{147}$$

for $k, k_1, k_2 \geq 0$.

A.1 inverses of π projection

$$\pi^{-1}([0]^T) = \{(+)[0]^T, (+)[4]^T\} \tag{148}$$

$$\pi^{-1}([1^a]^T) = (+)[1^a]^T$$
$$(+)[3,1^{a-1}]^T$$
$$(-)[4,2,1^{a-2}]^T$$
$$(-)[4,4,1^{a-2}]^T \quad (149)$$

$$\pi^{-1}([2,1^a]^T) = (+)[2,1^a]^T$$
$$(-)[4,3,1^{a-1}]^T \quad (150)$$

$$\pi^{-1}([2,2,1^a]^T) = (+)[2,2,1^a]^T$$
$$(-)[3,3,1^a]^T \quad (151)$$

$$\pi^{-1}([2^{a_2+3},1^{a_1}]^T) = (+)[2^{a_2+3},1^{a_1}]^T$$
$$(-)[3,3,2^{a_2+1},1^{a_1}]^T$$
$$(+)[4,3,3,2^{a_2},1^{a_1}]^T$$
$$(-)[4,4,4,2^{a_2},1^{a_1}]^T \quad (152)$$

for $a, a_1, a_2 \geq 0$.

B $\tilde{\pi}$ projection

The non-zero $\tilde{\pi}$ projections are those that "make sense"

$$\tilde{\pi}([2^{k_2},1^{k_1}]^T) = [2^{k_2},1^{k_1}]^T$$
$$\tilde{\pi}([3,1^k]^T) = [1^{k+1}]^T$$
$$\tilde{\pi}([4]^T) = [0]^T \equiv \mathbf{1} \quad (153)$$

B.1 Inverses of $\tilde{\pi}$ projection

$$\tilde{\pi}^{-1}([0]^T) = [0]^T$$
$$[4]^T \quad (154)$$

$$\tilde{\pi}^{-1}([1^a]^T) = [1^a]^T$$
$$[3,1^{a-1}]^T \quad (155)$$

$$\tilde{\pi}^{-1}([2^b,1^c]^T) = [2^b,1^c]^T \quad (156)$$

for $a, b \geq 1, c \geq 0$.

C Clebsch-Gordan identities

C.1 $V^{\otimes(n_1+n_2)}$

Suppose we have a decomposition of the fundamental V of $GL(M)$

$$V^{\otimes n} = \bigoplus_{\Lambda \in P(n,M)} V_\Lambda^{S_n} \otimes V_\Lambda^{GL(M)} \quad (157)$$

with Clebsch-Gordan

$$C^{\mu_1 \cdots \mu_n}_{\Lambda, M_\Lambda, a_\Lambda} \quad (158)$$

Suppose we want to decompose this into $n = n_1 + n_2$

$$V^{\otimes n} = V^{\otimes n_1} \otimes V^{\otimes n_2} = \left(\bigoplus_{\Lambda_1 \in P(n_1,M)} V^{S_n}_{\Lambda_1} \otimes V^{GL(M)}_{\Lambda_1} \right) \otimes \left(\bigoplus_{\Lambda_2 \in P(n_2,M)} V^{S_n}_{\Lambda_2} \otimes V^{GL(M)}_{\Lambda_2} \right) \tag{159}$$

The Clebsch-Gordan coefficients are related by

$$C^{\mu_1 \cdots \mu_n}_{\Lambda, M_\Lambda, a_\Lambda} = \sum_{\Lambda_1,\Lambda_2} \sum_{a_{\Lambda_1}, a_{\Lambda_2}} \sum_{M_{\Lambda_1}, M_{\Lambda_2}} \sum_{\tau \in g(\Lambda_1,\Lambda_2;\Lambda)} C^{a_{\Lambda_1}, a_{\Lambda_2}}_{a_\Lambda, \tau} \, C^{M_{\Lambda_1}, M_{\Lambda_2}}_{M_\Lambda, \tau} \, C^{\mu_1 \cdots \mu_{n_1}}_{\Lambda_1, M_{\Lambda_1}, a_{\Lambda_1}} \, C^{\mu_{n_1+1} \cdots \mu_n}_{\Lambda_2, M_{\Lambda_2}, a_{\Lambda_2}} \tag{160}$$

$C^{M_{\Lambda_1}, M_{\Lambda_2}}_{M_\Lambda, \tau}$ is the $GL(M)$ Clebsch-Gordan; $C^{a_{\Lambda_1}, a_{\Lambda_2}}_{a_\Lambda, \tau}$ is the S_n outer product.

C.2 Sym($W^{\otimes k}$)

Consider Sym($W^{\otimes k}$) where $W = V_1 \otimes V_2$ and V_1 is the fundamental rep of $GL(M)$ and V_2 of $GL(M')$. A representative would be

$$A_{h_1 \mu_1} \cdots A_{h_k \mu_k} \tag{161}$$

where the $A_{h_i \mu_i}$ all commute.

We can consider W as the fundamental rep of $GL(MM')$ so that

$$\text{Sym}(W^{\otimes k}) = V^{GL(MM')}_{[k]} \tag{162}$$

The Clebsch-Gordan for this is

$$C^{h_1 \mu_1 \cdots h_k \mu_k}_{[k], M_{[k]}} \tag{163}$$

However, decomposing in terms of $GL(M)$ and $GL(M')$ separately we have

$$V_1^{\otimes k} = \bigoplus_{\Lambda_1 \in P(k,M)} V^{S_k}_{\Lambda_1} \otimes V^{GL(M)}_{\Lambda_1} \tag{164}$$

with Clebsch-Gordan coefficient

$$C^{h_1 \cdots h_k}_{\Lambda_1, M_{\Lambda_1}, a_{\Lambda_1}} \tag{165}$$

and

$$V_2^{\otimes k} = \bigoplus_{\Lambda_2 \in P(n,M')} V^{S_k}_{\Lambda_2} \otimes V^{GL(M')}_{\Lambda_2} \tag{166}$$

with Clebsch-Gordan coefficient

$$C^{\mu_1 \cdots \mu_k}_{\Lambda_2, M_{\Lambda_2}, m_{\Lambda_2}} \tag{167}$$

Given the S_k invariance of Sym($W^{\otimes k}$) we must have for the S_k inner product

$$[k] \in \Lambda_1 \otimes \Lambda_2 \tag{168}$$

which forces $\Lambda_1 = \Lambda_2$ and we must sum over the S_k states $a_{\Lambda_1} = a_{\Lambda_2}$. So that

$$\left| [k], M_{[k]} \right\rangle = C^{h_1 \mu_1 \cdots h_k \mu_k}_{[k], M_{[k]}} = \sum_{\Lambda_1} \sum_{a_{\Lambda_1}} C^{[k], M_{[k]}}_{\Lambda_1, M_{\Lambda_1}, M'_{\Lambda_1}} C^{h_1 \cdots h_k}_{\Lambda_1, M_{\Lambda_1}, a_{\Lambda_1}} C^{\mu_1 \cdots \mu_k}_{\Lambda_1, M'_{\Lambda_1}, a_{\Lambda_1}} \tag{169}$$

Counting-wise this is

$$\text{Dim}_{MM'}[k] = \sum_{\Lambda_1 \in P(k, \min(M,M'))} \text{Dim}_M \Lambda_1 \text{Dim}_{M'} \Lambda_1 \tag{170}$$

C.3 Sym$(W^{\otimes 2t+k'})$

We want to combine Appendix Sections C.2 and C.1. First we do the split

$$W^{\otimes 2t+k'} \to V_1^{\otimes 2t+k'} \otimes V_2^{\otimes 2t+k'} \tag{171}$$

so that

$$C^{h_1\mu_1\cdots h_k\mu_k}_{[k],M_{[k]}} = \sum_{K\in P(k,\min(M,M'))} \sum_{a_K} C^{h_1\cdots h_k}_{K,M_K,a_K} C^{\mu_1\cdots\mu_k}_{K,M'_K,a_K} \tag{172}$$

Then split each tensor into $k = 2t + k'$ according to Appendix Section C.1

$$\begin{aligned} C^{h_1\mu_1\cdots h_k\mu_k}_{[k],M_{[k]}} = & \sum_{K\in P(k,\min(M,M'))} \sum_{a_K} \\ & \sum_{K_1,K_2} \sum_{a_{K_1},a_{K_2}} \sum_{M_{K_1},M_{K_2}} \sum_{\tau\in g(K_1,K_2;K)} C^{a_{K_1},a_{K_2}}_{a_K,\tau}\; C^{M_{K_1},M_{K_2}}_{M_K,\tau}\; C^{h_1\cdots h_{2t}}_{K_1,M_{K_1},a_{K_1}}\; C^{h_{2t+1}\cdots h_k}_{K_2,M_{K_2},a_{K_2}} \\ & \sum_{\Lambda_1,\Lambda_2} \sum_{a_{\Lambda_1},a_{\Lambda_2}} \sum_{M_{\Lambda_1},M_{\Lambda_2}} \sum_{\tau'\in g(\Lambda_1,\Lambda_2;K)} C^{a_{\Lambda_1},a_{\Lambda_2}}_{a_K,\tau'}\; C^{M_{\Lambda_1},M_{\Lambda_2}}_{M'_K,\tau'}\; C^{\mu_1\cdots\mu_{2t}}_{\Lambda_1,M_{\Lambda_1},a_{\Lambda_1}}\; C^{\mu_{2t+1}\cdots\mu_k}_{\Lambda_2,M_{\Lambda_2},a_{\Lambda_2}} \end{aligned} \tag{173}$$

Next we use a crucial branching coefficient identity

$$\sum_{a_K} C^{a_{K_1},a_{K_2}}_{a_K,\tau}\; C^{a_{\Lambda_1},a_{\Lambda_2}}_{a_K,\tau'} = \delta_{K_1\Lambda_1}\delta_{K_2\Lambda_2}\delta_{a_{K_1}a_{\Lambda_1}}\delta_{a_{K_2}a_{\Lambda_2}}\delta_{\tau\tau'} \tag{174}$$

which can be seen using bra-ket notation. This greatly simplifies our equation to

$$\begin{aligned} C^{h_1\mu_1\cdots h_k\mu_k}_{[k],M_{[k]}} = & \sum_{K\in P(k,\min(M,M'))} \sum_{K_1,K_2} \sum_{a_{K_1},a_{K_2}} \sum_{M_{K_1},M_{K_2}} \sum_{M'_{K_1},M'_{K_2}} \sum_{\tau\in g(K_1,K_2;K)} \\ & C^{M_{K_1},M_{K_2}}_{M_K,\tau}\; C^{h_1\cdots h_{2t}}_{K_1,M_{K_1},a_{K_1}}\; C^{h_{2t+1}\cdots h_k}_{K_2,M_{K_2},a_{K_2}} \\ & C^{M'_{K_1},M'_{K_2}}_{M'_K,\tau}\; C^{\mu_1\cdots\mu_{2t}}_{K_1,M'_{K_1},a_{K_1}}\; C^{\mu_{2t+1}\cdots\mu_k}_{K_2,M'_{K_2},a_{K_2}} \end{aligned} \tag{175}$$

References

[1] M. H. Rosas, "The Kronecker product of Schur functions indexed by two-row shapes or hook shapes," [arXiv:math/0001084].

[2] K. Koike and I. Terada, "Young Diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n," Jour. of Algebra. **107**, 466.