Complex Matrix Model Duality

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Moduli space of punctured Riemann surfaces from graphs

Theorem thanks to Mumford, Strebel, Harer and Penner in 80’s:

**Moduli space** of genus $g$ Riemann surfaces with $s$ punctures decorated by boundary lengths $\mathcal{M}_{g,s} \times \mathbb{R}^s_+$ has a **cell decomposition** given by inequivalent ribbon graphs with $s$ faces and lengths assigned to each edge.

Top-dim cells $\dim_{\mathbb{R}} = 6g - 6 + 3s$ come from edge-lengths of graphs with 3-valent vertices; lower-dim cells from collapsing edges to get higher-valency vertices.

E.g. for the thrice-punctured sphere $\mathcal{M}_{0,3} \times \mathbb{R}^3_+$ have graphs with 3 faces:

![Graphs](image)
Including closed string operators: the Kontsevich model

Generating function of correlation functions in 2d topological gravity given by Kontsevich matrix model

\[
\exp \sum_{g=0}^{\infty} \left\langle \exp \sum_{k} t_k \mathcal{O}_k \right\rangle_g = \rho(\Lambda)^{-1} \int [dM]_{n \times n}^{H} e^{-\frac{1}{2} \text{tr}(\Lambda M^2) + \frac{i}{6} \text{tr}(M^3)}
\]

The coupling coefficients are encoded in the constant matrix \( \Lambda \)

\[
t_k = -\sum_{i=1}^{n} \frac{1}{k \lambda_i^k} = -\frac{1}{k} \text{tr}(\Lambda^{-k})
\]

The colour index for each face gets associated to eigenvalues \( \lambda_i \) and hence to the couplings \( t_k \). The propagator can be transformed into an integral over the corresponding edge length \( p \) using the Schwinger trick

\[
\left\langle M^i_j \ M^k_l \right\rangle = \delta^i_j \delta^k_l \frac{2}{\lambda_i + \lambda_j} = 2 \delta^i_j \delta^k_l \int_0^{\infty} dp \ e^{-p(\lambda_i + \lambda_j)}
\]
The $\mathbb{C}Z$ matrix model

The $Z$ model for an $N \times N \mathbb{C}$ matrix ($N$ large but finite) is

$$Z(\{t\}, \{\bar{t}\}) = \int [dZ]_{N \times N}^\mathbb{C} e^{-\text{tr}(Z Z^\dagger) + \sum_{k=1}^{\infty} t_k \text{tr}(Z^k) + \sum_{k=1}^{\infty} \bar{t}_k \text{tr}(Z^{\dagger k})}$$

It generates the full genus expansions of

- Extremal correlation functions of certain half-BPS local operators for free $4d, \mathcal{N} = 4$ super Yang-Mills with gauge group $U(N)$.
- Scattering of integer-momentum tachyons in $c = 1$ string at self-dual radius, cosmological constant $\mu = iN$. The map to tachyons is $T_k \rightarrow \text{tr}(Z^k)$ and $T_{-k} \rightarrow \text{tr}(Z^{\dagger k})$.

The individual correlation functions are

$$\left\langle \text{tr}(Z^{k_1}) \cdots \text{tr}(Z^{k_p}) \text{tr}(Z^{\dagger k_1}) \cdots \text{tr}(Z^{\dagger k_q}) \right\rangle$$

How do we rewrite this model so that closed string insertions associate with faces rather than vertices?
The dual $\mathbb{C} F$ matrix model

The dual $F$ model is the same function of the couplings $\{t\}, \{\bar{t}\}$

$$Z(\{t\}, \{\bar{t}\}) = \int [dF]_{n \times n}^{\mathbb{C}} e^{-\text{tr}(AFBF^\dagger) + N \sum_{k=1}^{\infty} \frac{1}{k} \text{tr}[(FF^\dagger)^k]}$$

The couplings $\{t\}$ and $\{\bar{t}\}$ are encoded in constant matrices $A, B$

$$t_k = \sum_{i=1}^{n} \frac{1}{ka_i^k} = \frac{1}{k} \text{tr} A^{-k} \quad \bar{t}_k = \sum_{j=1}^{n} \frac{1}{kb_j^k} = \frac{1}{k} \text{tr} B^{-k}$$

The colour index for each face of the $F$ model Feynman diagrams comes with either $a_i$’s or $b_j$’s, so the couplings $\{t\}$ and $\{\bar{t}\}$ are associated to faces (dual to vertices of $Z$ model)

$$\langle F_j^i F_i^k \rangle = \frac{\delta_i^i \delta_j^k}{(a_i b_j - N)}$$
\[
\int [dZ] \ e^{- \text{tr}(ZZ^\dagger)} \ \frac{1}{\text{det}(A \otimes I_N - I_n \otimes Z)} \ \frac{1}{\text{det}(B \otimes I_N - I_n \otimes Z^\dagger)}
\]

\[
= \int [dZ, C, D] \ e^{- \text{tr}[ZZ^\dagger + C^\dagger AC - C^\dagger ZC + D^\dagger BD - D^\dagger Z^\dagger D]}
\]

\[
= \int [dC, D] \ e^{- \text{tr}[C^\dagger AC + D^\dagger BD - CC^\dagger DD^\dagger]}
\]

\[
= \int [dF, C, D] \ e^{- \text{tr}[FF^\dagger - D^\dagger F^\dagger C - C^\dagger FD + C^\dagger AC + D^\dagger BD]}
\]

\[
= \int [dF] \ e^{- \text{tr}(AFBF^\dagger)} + N \sum_{k=1}^\infty \frac{1}{k} \text{tr}[(FF^\dagger)^k]
\]
Example in action
Calculation: two-point function on the torus

$Z$ model graph is solid line; dual $F$ model graph is dashed line

\[
\langle \text{tr}(FF\dagger FF\dagger FF\dagger) \rangle_{\text{torus}} + \langle \text{tr}(FF\dagger FF\dagger) \text{tr}(FF\dagger FF\dagger) \rangle_{\text{torus}}
\]
\[=
\sum_{k=3}^{\infty} t_k \bar{t}_k k \left[ \begin{pmatrix} k \\ 3 \end{pmatrix} + \begin{pmatrix} k \\ 4 \end{pmatrix} \right] N^{k-2}
\]
\[=
\sum_{k=3}^{\infty} t_k \bar{t}_k \langle \text{tr}(Z^k) \text{tr}(Z^{\dagger k}) \rangle_{\text{torus}}
\]
Discrete Schwinger parameterisation of $F$ matrix model

Following analysis of Chekhov-Makeenko model (the Hermitian version of the $F$ model), rewrite $a_i = \sqrt{N} e^{\varepsilon l_i}$ and $b_j = \sqrt{N} e^{\varepsilon m_j}$ with discretisation parameter $\varepsilon$. The propagator becomes

$$\frac{1}{a_i b_j - N} = \frac{1}{N e^{\varepsilon l_i} e^{\varepsilon m_j} - N} = \frac{1}{N} \sum_{p=1}^{\infty} e^{-p\varepsilon (l_i + m_j)}$$

The sum is a discrete Schwinger parameterisation of the edge length for the propagator. Each summand comes from an edge of integer length $p\varepsilon$.

Edge lengths correspond to coordinates on (subspaces of) the moduli space of Riemann surfaces, so $F$ model correlation functions localise on discrete points in the moduli space. For each dual $Z$ correlation function there are furthermore only a finite number of points.

[This is no surprise since the correlation functions count very particular holomorphic Belyi maps from algebraic Riemann surfaces to $\mathbb{CP}^1$, whose Strebel differentials have integer-length critical graphs.]
Conclusions

- Same complex matrix model (Z model) generates tachyon scattering for \( c = 1, R = 1 \) string and computes correlation functions of half-BPS operators in free \( d = 4, \mathcal{N} = 4 \) SYM. Closed string insertions appear as vertices of the Feynman diagrams.

- There is a dual complex matrix model (F model) where now closed string insertions are associated to faces of the dual diagrams.

- Using a discrete Schwinger parameterisation of the F model propagators, the correlation functions localise on discrete points in (subspaces of) the moduli space of Riemann surfaces.

Questions:

- What is the geometric relation between the \( c = 1, R = 1 \) string and this sector of (small radius) \( AdS_5/CFT_4 \)?

- How does the dual worldsheet theory localise on the moduli space?

- Can correlation functions of other operators of (free) \( \mathcal{N} = 4 \) SYM be treated this way to gain a microscopic understanding of AdS/CFT?